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DERIVATIONS AND SIMULATIONS OF EVOLUTION EQUATIONS OF WAVY FILM FLOWS

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Abstract:

Wavy flows of viscous films on solid surfaces are considered. The focus is on approximate descriptions which are hinged on a single evolution equation. Perturbative approaches to constructing such theories are discussed. For several film flows, evolution equations obtained--along with the validity conditions of those theories--with the multiparametric perturbation approach are reviewed. The results of their three-dimensional numerical simulations on extended spatial intervals are discussed. Some unresolved fundamental questions concerning such film-flow studies are posed and discussed.

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foundations of the film flow research

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Introduction: Film flows and evolution equations

Our subject matter is the flow of liquid films on solid surfaces. The force driving the flow is usually gravity and/or an externally applied pressure. One boundary surface of the liquid layer is its interface with the supporting solid, the other a *fluid interface*. If the ambient fluid is a dynamically passive gas, the film has a *free surface*--as in flows down inclined planes or vertical cylinders. Otherwise, the film has a *fluid-fluid interface* with the surrounding active fluid. Such is the case in the so-called core-annular flow (see e.g. [51]), which is a two-fluid flow in a pipe.

One encounters such film flows in nature and in everyday life. Controllable flows of films are used in industry. Naturally, they have been studied for quite some time (see, for example, [69, 52, 71, 70, 72] for a considerable history of the film flow investigation). Currently, they continue to be a subject of growing research activity (see e.g. [15] for a recent review of work on a film flow down a vertical wall). Of course, *in principle* the film dynamics is known: it is given by the familiar Navier-Stokes (NS) equations [supplemented with appropriate boundary conditions (BC)]. Usually, however, one is interested less in the equations per se than in the dynamical fields--velocities, pressures, the position of the interface, etc.--which are unknown *solutions* of the NS equations. These are far from being easy to solve: indeed, one has to deal with a system of several coupled partial differential equations (PDE) which is additionally complicated with the moving boundary--whose dynamical PDE itself involves unknown fields, the fluid velocities. Even with the most powerful modern computers, such a full NS problem is too difficult to solve, especially--as is frequently needed--in *extended* space-time domains. Therefore, one would like to have a simpler, solvable description of evolution, provided that it yields a sufficiently good approximation to the exact solution of the NS problem. Ideally, each such simplified dynamical system should be accompanied by *validity conditions*, such that if the flow parameters satisfy these conditions, the solutions are guaranteed to be good approximations to the corresponding exact NS evolutions.

The most favorable case of such a simplification is the one in which the problem reduces to a *single* PDE--which as a rule is the one for the *thickness* of the film. If the solution is found, the theory yields the velocities and pressures as explicit expressions in terms of the film thickness. However, even if one succeeds in obtaining such a simplified *evolution equation* (EE), it still cannot, in practice, be solved analytically; so one turns to computers. Over a number of years, we engaged in such mathematical modeling and numerical simulations of film flows. Below, we discuss some particular problems we considered, methods which have been developed, the results obtained, and also some difficult questions which have not yet been answered. (Some of the results presented here have not been published before.) The work of other researchers is touched upon only inasmuch as it was judged relevant to those issues. Accordingly, our reference list is far from being comprehensive. We sincerely apologize to those colleagues whose relevant work may have been inadvertently overlooked here.

The rest of this paper is organized as follows. In the next section, we introduce our *multiparametric perturbation* (MP) approach and discuss its advantages over previous perturbative methods. In section 3, we discuss the resulting evolution equations and conditions of their validity obtained with this approach, as well as the results of numerical simulations of the evolution equations for various film flows. In section 4, some unanswered fundamental questions pertaining to these film-flow studies are discussed. The last section summarizes the paper.

Perturbation approaches

- [*An example of the less-formal MP approach at work: 2-D vertical film flow*](#)
- [*The leading two orders of the fully formal MP approach: the strongly dispersive case*](#)
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An example of the less-formal MP approach at work: 2-D vertical film flow

We consider a layer, of an average thickness \bar{h}_0 (the overbar here and below indicates a *dimensional* quantity), of an incompressible Newtonian liquid--of a density $\bar{\rho}$, viscosity $\bar{\mu}$, and surface tension $\bar{\sigma}$ --which flows under the action of gravity (whose acceleration is denoted \bar{g}) down a *vertical* solid plane (also called "the wall". Here we confine ourselves to this particular case of an *inclined* film for the sake of simplicity, and only consider *two-dimensional* (2-D) wavy flows. Some more general results are given in section 3.1; we refer the reader to [48] for more results and the theory for the general case, a 3-D flow down an inclined plane.). There is a well-known time-independent, "Nusselt's", solution of the NS problem for the vertical film. The thickness of the Nusselt film is constant (hence, Nusselt's flow is also referred to as a "flat-film" solution). The only nonzero component of velocity is the downward one. It only changes across the film, starting from the zero value at the wall. The free-surface value \bar{U}_v of the Nusselt velocity is $\bar{U}_v = \bar{g}\bar{h}_0^2/(2\bar{\nu})$ (where $\bar{\nu} = \bar{\mu}/\bar{\rho}$ is the kinematic viscosity).

We nondimensionalize all quantities with units based on $\bar{\rho}$, \bar{h}_0 , and \bar{U}_v . (As we will see below, exactly two independent *basic parameters* appear in the *dimensionless* equations and boundary conditions; one can choose e.g. the Reynolds number $R = \bar{h}_0\bar{U}_v/\bar{\nu}$, and the Weber number $W = \sigma R/2$ as such basic parameters.)

The x axis of our system of coordinates is normal to the solid plane and directed away from it; the y axis is in the spanwise (i.e. horizontal and parallel to the wall) direction; and the z axis is directed downward. [A subscript x, y, z , or (time) t , will always--whether preceded by a comma or not--indicate the differentiation with respect to that variable.] For simplicity (as was mentioned above), we first consider the 2-D flow, so that v , the y component of velocity, is zero. The x and z components of velocity are denoted, respectively, u and w .

In a coordinate system moving (with respect to the laboratory reference frame) with a velocity V in the z -direction, the NS equations (for velocities measured in the *laboratory* frame) written in the dimensionless form are (see e.g. [2])

$$u_t + uu_x + ww_x - Vu_z = -p_x + (u_{xx} + u_{zz})/R, \quad (1)$$

$$w_t + uw_x + ww_z - Vw_z = -p_z + 2/R + (w_{xx} + w_{zz})/R. \quad (2)$$

The continuity equation is

$$u_x + w_z = 0. \quad (3)$$

The boundary conditions are as follows. The no-slip conditions at the solid surface are

$$u = w = 0 \quad \text{at} \quad x = 0. \quad (4)$$

The tangential-stress balance condition at the free surface is

$$p_{11}h_z + p_{13}[1 - h_z^2] - p_{33}h_z = 0 \quad (5)$$

and the normal-stress one is

$$[p_{11} + p_{33}h_z^2 - 2p_{13}h_z][1 + h_z^2]^{1/2} = \sigma h_{zz}. \quad (6)$$

Here, the stress components p_{ij} are

$$p_{11} = -p + 2u_x/R, \quad p_{33} = -p + 2w_z/R, \quad p_{13} = (u_z + w_x)/R. \quad (7)$$

Finally, the kinematic condition at the free surface is

$$h_t + wh_z - Vh_z = u. \quad (8)$$

Let w_0 denote the Nusselt velocity for an imaginary film of *constant* thickness h which is equal to the local thickness $h(z,t)$ of the real film (which is in agreement with experiments [1]): $w_0 = 2hx - x^2$. [Note that w_0 depends not only on x but also on z and t --through $h(z,t)$.] The *reference* normal component of velocity, u_0 , is chosen to satisfy incompressibility (3); i.e., it is defined as the solution of $u_{0x} = -w_{0z}$, with the no-slip boundary condition $u_0 = 0$ at $x = 0$. This solution clearly is $u_0 = -x^2 h_z$. The above NS problem becomes an exact one for a new set of unknowns-- h, u_1, w_1 , and p --by the substitution $u = u_0 + u_1$ and $w = w_0 + w_1$ (the remaining unknowns p and h are not changed). For example, the kinematic boundary condition at the free surface $x = h$ [Eq. (8)] becomes

$$h_t + (w_0 + w_1 - V)h_z - u_0 - u_1 = 0. \quad (9)$$

If one looks into derivations of known evolution equations (EEs), one realizes that each of them is invariably obtained from exactly a kinematic condition of the type (9), which can be done in the following way. First, the NS equations are simplified by discarding some terms, to arrive at essentially *ordinary* differential equations (ODEs) for the velocities and pressure as functions of the streamwise coordinate x [with the other independent variables entering *as parameters only*, through $h(z,t)$]. It is straightforward to solve these simple (albeit nonhomogeneous) ODEs, since their coefficients are constant. Then, one proceeds to substitute the resulting solutions for velocities (in terms of h) into the kinematic condition. This yields a closed PDE for h , i.e. the evolution equation. In our derivation, we follow this recipe as well, but we take care to discard *only those* terms in the NS equations which *must* be dropped if one is to obtain *solvable* ODEs.

Since u_1, w_1 , and p are to be found by reducing the NS equations to an ODE *in* x for each of these dependent variables, it is clear that the equation for p is the (reduced) x -NS equation: indeed, this is the only equation (of the full NS problem) containing a derivative of p with respect to x . Consequently, u_1 has to be determined from the incompressibility condition (since it contains u_{1x}), which can be done after finding w_1 from the z -NS equation. The order in which these equations are solved can, in principle, play a role, since in solving for a particular variable, one has to neglect the terms containing those of the other variables which have not been determined prior to that. After a little investigation, one can see that the most natural order is as follows.

First, one finds p from the x -NS equation, with the BC coming from the normal-stress balance condition at the free surface [for simplicity, the pressure of the ambient gas has been neglected in the formulation (1-8)]. In this problem for p , one clearly has to discard all terms containing one or more of the unknowns u_1 and w_1 . Next, the z -NS equation is recast into an ODE for w_1 . Here, one has to discard only those terms containing u_1 ; the p -term can be retained, as it is now a *known* expression in terms of h . But all terms containing w_1 , except for the viscous one with w_{1xx} , must be discarded as well-- otherwise one does not get a constant-coefficient ODE in x .

Let us denote by A the characteristic amplitude of the surface deviation $\eta \equiv h - 1$, and let T and L be, respectively,

the characteristic time- and length-scales. Then, for example, from the condition that the *neglected* viscous term should be smaller than the one retained, $w_{1zz} \ll w_{1xx}$, after estimating these quantities in terms of the characteristic scales, one arrives at the requirement $(1 + A)^2/L^2 \ll 1$. This is the small-slope condition. Usually, it is rather *postulated* in the "lubrication" or "long-wave" derivations, but here this condition arises as a consequence of our "derivability principle". In obtaining this condition, we have used the following natural estimates of the derivatives: for the x -derivatives, we have $\partial/\partial x \sim 1/(1 + A)$, in the sense that $\partial/\partial x \sim 1$ if $A \sim 1$ or $A \ll 1$, and $\partial/\partial x \sim 1/A$ if $A \gg 1$ (indeed, $h \sim A$ for $A \gg 1$; and the velocities change from being zero at $x=0$ to their full magnitudes at $x=h$; hence, the characteristic lengthscale of change is h , which is $\sim A$). Also, $\partial/\partial t \sim 1/T$, $\partial/\partial z \sim 1/L$, etc. One can see that, similar to the small-slope condition above, the condition $(R/T)(1 + A)^2 \ll 1$ follows; etc. In fact, one can estimate the velocities even before solving the equations. Substituting such estimates into inequalities demanding the smallness of terms which must be discarded, one finally arrives at the set of independent conditions for the derivation to be justified. These validity conditions can be combined in the following form:

$$\max \left[\frac{(1 + A)W}{L^3}, \frac{(1 + A)^4 R}{L}, \frac{(1 + A)^2}{L^2} \right] \ll 1. \quad (10)$$

[One notes that all three of these conditions can be obtained e.g. already from the requirement of negligibility of the term containing u_{1z} in the BC for w_1 , Eq. (5).] Due to these conditions, even some terms which can be retained in the equations--and handled, in principle, without any difficulty--can be shown to be actually negligible. Also, some other terms are estimated to lead to a negligible contribution to the final evolution equation and therefore can be discarded as well. As a result, one can see that the expression for p is in fact the solution of the problem [see Eq. (1) and (6)]

$$p_x = [u_{0xx}/R =] - 2h_z/R,$$

$$p = [2u_{0x}/R - \sigma h_{zz} =] - 4hh_z/R - \sigma h_{zz} \quad (x = h),$$

that is

$$p = -2(x + h)h_z/R - \sigma h_{zz}. \quad (11)$$

Note that this pressure contains a *viscous* contribution (the first term on the right-hand-side) in addition to the usual surface-tension part.

The equation for w_1 is the (simplified) z -NS equation:

$$w_{1xx} = R(u_0 w_{0x} + w_0 w_{0z} - 2w_{0z} + p_z + w_{0t}) - w_{0zz}.$$

(Note that we use the system of coordinates of a *moving* reference frame, whose z -directed velocity with respect to the solid plane is $V=2$ --that is equal to the well-known phase velocity of the infinitesimal waves; all the velocities, however, are measured with respect to the *laboratory* frame.) We also have to satisfy the following boundary conditions: first, $w_1 = 0$ at $x = 0$; and second, from the tangential-stress balance equation at $x=h$ (5), we have

$w_{1x} = -u_{0z} - 2u_{0x}h_z + 2w_{0z}h_z$. The solution of this problem is

$$w_1 = \frac{x^4}{6} R h h_z - \frac{2x^3}{3} [h_{zz} + R h_z] - x^2 [h h_{zz} + W h_{zzz} + (h_z)^2]$$

$$+x \left[5h^2 h_{zz} + 2W h h_{zzz} + 2R h^2 h_z - \frac{2}{3} R h^4 h_z + 10h(h_z)^2 \right]. \quad (12)$$

Next, the equation for u_1 derives from the continuity equation: $u_{1x} = -w_{1z}$, where w_1 is already known (12); and the no-slip BC requires that $u_1 = 0$ at $x = 0$. The solution of this problem is

$$u_1 = -\frac{x^5}{30} R(h h_z)_z + \frac{x^4}{6} [h_{zzz} + R h_{zz}] + \frac{x^3}{3} \left[\frac{1}{2} (h^2)_{zzz} + W h_{zzzz} \right] - \frac{x^2}{2} \left\{ 5(h^2 h_{zz})_z + 2W [(h h_{zzz})_z] + 2R (h^2 h_z)_z - \frac{2}{3} R (h^4 h_z)_z + 10(h h_z^2)_z \right\}. \quad (13)$$

One substitutes the expressions (12) and (13) for velocities in terms of h (taken at $x=h$) into the kinematic condition (9), which yields the EE

$$h_t - \frac{R}{12} (h^5)_{tz} + 2(h^2 - 1)h_z + R \left[\frac{5}{6} h^4 h_z - \frac{3}{10} h^6 h_z \right]_z + \frac{2}{3} [h^3 W h_{zzz}]_z + 2h^4 h_{zzz} = 0. \quad (14)$$

We have omitted all of the several dispersive terms except for a single one--the last term on the left-hand-side (l.h.s.) of Eq. (29)--since in the limit of small deviations η they turn out to be negligible [see Eq. (15) below]; and if the amplitude of η is not small, the equation, in any case, can only be good as a *qualitative model*, as we will also discuss below. The solutions of this equation yield a good approximation to the true evolutions of the thickness h --for some time, at least--*as long as* the conditions (10) are satisfied.

Using those conditions and estimating all of the members of the EE (29) in terms of the current amplitude A , the (current) lengthscale L , and the basic parameters, one can see the following facts: If A is *not* small, the equation can be written as simply $h_t + 2(h^2 - 1)h_z = 0$ (or just $h_t + 2h^2 h_z = 0$ in the laboratory reference frame). Indeed, all other terms of the EE (29) turn out to be much smaller than the two selected into the shortened equation above. The solutions of this short equation (called "the simplest hyperbolic equation" in [96]) are well known to exhibit *steepening* of the wavefronts that leads to *breaking* of the waves in a finite time. It follows that the EE (29) cannot be valid *globally*, i.e. for all time. But one can easily see that this EE is equivalent to the well-known Benney equation: The former transforms into the latter (in the laboratory reference frame and with the dispersion term omitted) by the substitution of the term $-2h^2 h_z$ in place of h_t (this trick was called the "trade of time- for space-derivative" in [2])

into the second term of (29). Hence, the large-amplitude waves can be approximated by the Benney equation *at best* for a finite time--a fact that was established before in [35]. Thus, here from a different direction, we arrive at the same conclusion: a (film thickness) evolution equation which would provide a good time-uniform approximation to *large-amplitude* waves on films flowing down vertical (as well as inclined, as is discussed below) planes, *does not exist*. (This is in contrast to films flowing down vertical *cylinders*, where such an equation *does* exist: it has been obtained in [34].) For a *quantitatively* good approximation of such large-wave regimes of planar films, one has to be content with a less drastic simplification of the original NS problem, which would contain a *system* of at least two coupled PDEs--such as those studied e.g. by Chang, Demekhin, and their co-workers (see e.g. [17, 16]). The domain of validity of such evolution *systems* may clearly be larger than that of theories hinged on a *single* evolution equation; however, the difficulty faced in attempting to solve them is correspondingly more severe, and indeed can approach the formidable difficulty of the original NS system.

[Note that the two equations, (29) and the Benney one, are not necessarily equivalent when the validity conditions (10)

break down. In fact, there are indications that our equation might make a better model; we will discuss this point in section 3.1.1.]

The case of small amplitude waves, $(\eta \sim) A \ll 1$, is different: here, one does have a theory hinged on a *single* evolution equation which nevertheless can yield a *quantitatively* good approximation. Indeed, by substituting $h = 1 + \eta$ into the general EE (29), one gets

$$\eta_t + 4\eta\eta_z + \frac{8}{15}R\eta_{zz} + \frac{2}{3}W\eta_{zzzz} + 2\eta_{zzz} = 0, \quad (15)$$

the evolution equation for the small-amplitude waves. This equation has appeared before in [93]. [It is, however, a particular case of the more general equation (31) below (which was derived in [48, 47]) that allows for an arbitrary inclination of the film plane to the horizontal and also for the dependence of waves on the spanwise coordinate y .]

We use the inequality $A \ll 1$ to transform the general conditions (10) and obtain the *local-validity* conditions for the small-amplitude EE (15):

$$\max(W/L^3, R/L, 1/L^2, A) \ll 1. \quad (16)$$

We call these conditions local (in time) since they involve the parameters L and A which, in contrast to the basic parameters R and W , can *change* with time. Indeed, due to the dissipativeness of the EE (15), the system evolves towards an attractor and thus essentially forgets the initial conditions. On the attractor, there can be fluctuations, but no systematic change in time is possible. Then, following the ideas of [6, 37], the second-derivative term of the Eq. (15)--which term is well known to lead to the *growth* of disturbances--should be of the same order of magnitude as the stabilizing, fourth-derivative one (while the third-derivative term is purely *dispersive*: it makes only a purely imaginary contribution to the linear-theory growth rate). Hence, the (dimensionless) characteristic lengthscale *at large times*, L_a , is estimated to be

$$L_a = [W/(4R/5)]^{1/2}.$$

Similarly, the asymptotic magnitude of the characteristic amplitude A_a is determined by the balance between the nonlinear "convective" term and either the dispersive term or the capillary one (whichever is larger): $A_a = \max(W/L_a^3, 1/L_a^2)$. Using these estimates, the condition (16) can be written as

$\max(W/L_a^3, R/L_a, 1/L_a^2) \ll 1$. Noting that $W/L_a^3 = (4R/5)/L_a$, we finally can write this as $\max(R/L_a, L_a^{-2}) \ll 1$. Thus the validity conditions are

$$A^2 \equiv 1/L_a^2 \ll 1 \quad \text{and} \quad R_1 \equiv R/L_a \ll 1. \quad (17)$$

These conditions involve only the basic (time-independent) parameters of the flow. If the basic parameters satisfy these conditions, the EE (15) is valid (yields a good approximation) *for all time*. Hence, these conditions may be termed the *global-validity* condition.

We can transform Eq. (15) to a "canonical" form--which would contain a minimum of "tunable" constants--by rescaling $\eta = N\tilde{\eta}$, $z = L_a\tilde{z}$, and $t = T_a\tilde{t}$. We take $N = W/(6L_a^3)$ and $T_a = (3W/2)/(4R/5)^2$.

Dropping the tildes in the notations of variables, the resulting canonical form of the small-amplitude evolution equation is

$$\eta_t + \eta\eta_z + \eta_{zz} + \eta_{zzzz} + \lambda\eta_{zzz} = 0, \quad (18)$$

where $\lambda = 3/\sqrt{W(4R/5)}$.

The way we have derived the above theory is in fact a refinement of the less-formal version of the *multiparametric perturbation* (MP) approach used before in [[6](#), [31](#), [34](#), [37](#)].

The leading two orders of the fully formal MP approach: the strongly dispersive case

In the more formal version of the method (see [31, 32, 34]), one represents the variables of the problem as multiple series in powers of *two* (or more in other physical problems) independent perturbation parameters (this is why we call the method *multiparametric*). The choice of the appropriate perturbation parameters depends on the magnitude of the *dispersion constant* λ in Eq. (18):

(i) If $\lambda \sim 1$, the appropriate small parameters are Λ^2 and R_1 of Eq. (17). Then the evolution equation and the approximate expressions for velocities and pressure follow from the *leading-order* equations [which couple together the leading-order coefficients of the (Λ^2, R_1) -power series of the unknowns]. There is no need for the more formal procedure unless one is concerned with higher-order corrections to the leading-order results.

(ii) When $\lambda \ll 1$, the appropriate independent small parameters are Λ^2 and λ : It follows that $R_1 \ll 1$ since one finds $R_1 \propto \Lambda^2 \lambda$, with the proportionality coefficient being ~ 1 . In this case the (higher-order of magnitude) dispersive term in Eq. (18) can be neglected, leaving behind the KS equation as a result. The formal MP procedure yields the KS equation at the leading order.

(iii) Finally, in the case $\lambda \gg 1$, we introduce

$$\varepsilon \equiv 1/\lambda \quad (\varepsilon \ll 1)$$

and

$$\Lambda \equiv \sqrt{\Lambda^2} \quad (\Lambda \ll 1).$$

It is easy to see that in terms of the independent small parameters ε and Λ , other basic parameters are given by the following expressions: $R = R_0 \Lambda \varepsilon$, $W = W_0 \Lambda^{-1} \varepsilon$, and $\sigma (= 2W/R) = \sigma_0 \Lambda^{-2}$, where $R_0 = 15/4$,

$W_0 = 3$, and $\sigma_0 = 8/5$.

The EE (18) can be rescaled to the form

$$\eta_t + \eta \eta_z + \varepsilon (\eta_{zz} + \eta_{zzzz}) + \eta_{zzz} = 0. \quad (19)$$

The leading order equation--which follows by neglecting the terms multiplied by ε --is the Korteweg-de Vries (KdV) one,

$$\eta_t + \eta \eta_z + \eta_{zzz} = 0. \quad (20)$$

However, as was shown some time ago [55, 31], the dissipative terms of Eq. (19), although they are of a higher order, nevertheless play an important role. Namely, it is well known that there is a one-parameter family of *soliton* solutions to the KdV equation, and the smaller-amplitude solitons have a greater width. If one starts with a wide soliton as an initial condition to Eq. (19), the stabilizing fourth-derivative term is much smaller than the destabilizing second-

derivative one, and as a result the amplitude of the soliton will slowly *grow*. The soliton width quickly adjusts to the changing amplitude. The width continues to decrease in this manner until the two dissipative terms of (19) are finally in balance. Thus, these small terms determine the final width, and hence the amplitude and the velocity, of the soliton solution.

Hence, one needs to employ the formal version of the MP approach in order to determine if the higher-order dissipative terms of Eq. (19) have been found correctly [the answer will be seen to be affirmative in the case under consideration; however, in some cases of the *inclined*-film problem the formal analysis can reveal [61] certain *corrections* to the coefficients obtained with the less-formal version of the MP approach]. This formal MP procedure is as follows.

Power series

One starts by substituting into the NS problem (1)-(9) the power series

$$p = \Lambda^2 \varepsilon^{-1} \sum P_{ij} \Lambda^i \varepsilon^j, \quad (21)$$

$$w_1 = \Lambda^4 \sum W_{ij} \Lambda^i \varepsilon^j, \quad (22)$$

$$u_1 = \Lambda^5 \sum U_{ij} \Lambda^i \varepsilon^j, \quad (23)$$

$$h - 1 \equiv \eta = \Lambda^2 \sum H_{ij} \Lambda^i \varepsilon^j, \quad (24)$$

$$V = \sum V_{ij} \Lambda^i \varepsilon^j, \quad (25)$$

where $\sum \equiv \sum_{(i,j)=(0,0)}^{(\infty,\infty)}$. In these series, V_{ij} are constants, and the *coefficient functions* (CF)-- P_{ij} , W_{ij} , etc.-- depend on (i) x (except for H_{ij}), (ii) the rescaled downstream coordinate ξ , such that

$$\frac{\partial}{\partial z} = \Lambda \frac{\partial}{\partial \xi},$$

and (iii) on a sequence of time-variables of different scales-- τ_{00} , τ_{01} , etc.--such that

$$\frac{\partial}{\partial t} = \Lambda^3 \sum \Lambda^i \varepsilon^j \frac{\partial}{\partial \tau_{ij}}.$$

The exponents of Λ and ε in front of sums in these series are determined from the results of the preceding less-formal MP analysis (similar to e.g. [32, 34]). The series for the reference velocities are easily determined from the expressions $u_0 = -\Lambda x^2 \eta_\xi$ and $w_0 = (2x - x^2) + 2x\eta$, by substituting into them the series (24) for η .

Each NS equation acquires the form in which the l.h.s is a (double) series in the powers of Λ and ε , and the r.h.s. is 0. Requiring that each coefficient of the l.h.s. is equal to zero separately, we obtain a sequence of problems for the coefficient functions. They can be solved one after another (the higher-order problems contain as their coefficients the CFs found as a result of solving the lower-order problems). Thus, the MP approach yields a theory which is "consistent to all orders in perturbations", in the terminology of Ref. [10].

Leading-order system

One can readily see that the kinematic condition (9) in the leading order Λ^3 only determines that $V_{00} = 2$. But from the Λ^5 -order of that equation, the leading-order EE is to come:

$$H_t + 4HH_\xi - U_{00}(x=1) = 0. \quad (26)$$

[(i) For brevity, we denote H_{00} as simply H and τ_{00} as τ . (ii) Note that the term $V_{10}H_{10\xi}$ appears in this equation; this term is eliminated by choosing $V_{10} = 0$.] To express the last member of this equation in terms of H , one proceeds through the equations of the NS problem, one after another.

Equation (1) in the order $\Lambda^2\varepsilon^{-1}$ yields a first-order ODE in x for P_{00} , viz. $P_{00x} = -(2/R_0)H_\xi$. The appropriate BC comes from the same order of the normal-stress Eq. (6): $P_{00}(x=1) = 2u_{0x}[2, -1]/R_0 = -4H_\xi/R_0$ (where a new type of notation has been used: $u_{0x}[2, -1]$ is the notation for the CF of $\Lambda^2\varepsilon^{-1}$ in the power-series expression of u_{0x} , etc.) The solution is $P_{00} = -(2/R_0)H_\xi(x+1)$.

Next, W_{00} is found from the z -NS equation (2) in order [3,-1], $W_{00xx} = -(4x+2)H_{\xi\xi}$. The BCs are (i) the no-slip Eq. (4) in order [4,0] and (ii) the tangential-stress Eq. (5) in order [3,-1]: $W_{00x}(x=1) = H_{\xi\xi}$. This yields the solution $W_{00} = -(\frac{2}{3}x^3 + x^2 - 5x)H_{\xi\xi}$. From the incompressibility, Eq. (3) in order [5,0], one finds $U_{00} = \frac{1}{6}(x^4 + 2x^3 - 15x^2)H_{\xi\xi\xi}$. Substituting this expression (evaluated at $x=1$) into Eq. (26), we obtain the leading-order EE:

$$H_t + 4HH_\xi + 2H_{\xi\xi\xi} = 0.$$

The ε -correction to the leading order

The ε -correction to the EE involves H_{01} and comes from the kinematic condition, Eq. (9), when taken one order in ε higher than before, viz. in the order [5,1]. Preliminary, one finds P_{01} , W_{01} , and U_{01} by going through the NS equations in the same sequence as above but taking each at the next order in ε ; e.g. the z -NS equation (2) is used in the order [3,0]. In the end, one arrives at the equation

$$H_{1\tau} + 4H_1H_\xi + 4HH_{1\xi} + 2H_{1\xi\xi\xi} + H_{\tau_1} - V_{21}H_\xi + \frac{8}{15}R_0H_{\xi\xi} + \frac{2}{3}W_0H_{\xi\xi\xi\xi} = 0 \quad (27)$$

(where we have simplified the notations H_{01} and τ_{01} to H_1 and τ_1 , respectively). This can be considered to be a nonhomogeneous equation for H_1 , and then the constant V_{21} can be tuned in order that the *solvability condition* on the forcing part (the sum of terms that do not contain H_1) be satisfied. (The exact form of that condition is determined only *after* the BC in ξ are stipulated; then, it can be obtained e.g. by multiplying the equation by H , integrating over the coordinate ξ and the fast time τ by parts, and using the BC.) We note that, if necessary (which does not seem to be the case in the present problem), one could use a more complicated arrangement--of the type which was indicated already in [34] and is used for the time t here--for the *coordinate* z as well, such that one would have

$$\frac{\partial}{\partial z} = \Lambda^3 \sum \Lambda^{i\varepsilon^j} \frac{\partial}{\partial \xi_{ij}}.$$

If the power series (24) for η is substituted directly into Eq. (9), the leading order [5,0] yields the KdV equation (20), and the higher order [5,1] leads to (19). This means that all the coefficients appearing in the equation (15) (including those of the--possibly, *small*--dissipative terms) are correct even in the case of *large* dispersion.

Earlier perturbation approaches and the problem of validity conditions

To discuss the advantages of the MP approach, we will consider other perturbation schemes used in derivations of evolution equations (see also section [4.1](#)).

- [Long-wave approach](#)
- [Single-parameter approach](#)

Long-wave approach

In early derivations (e.g. [2, 84, 75, 62, 41, 63]), starting from the pioneering work of Benney [11], the formal series were in powers of a single *local* "long-wave" parameter $\alpha \equiv 1/\bar{L}$, where L is a local streamwise lengthscale of the type introduced above. The principal unknown, the film thickness h , remains, at least initially, whole (i.e. without being expanded in the power series); and each basic parameter of the NS problem is assigned a certain order (e.g. the surface tension can be $\sigma \sim \alpha^{-2}$)--in most cases, just implied to be of order α^0 . The EE for h is obtained in the form of an infinite power series

$$N_0 h + \alpha N_1 h + \alpha^2 N_2 h + \dots = 0 \quad (28)$$

where $N_i (i = 0, 1, 2, \dots)$ are differential operators (containing the derivatives with respect to the temporal and spatial variables). Usually, the series is truncated to retain only the first two--or, less commonly, three--terms. If one thinks about the possible conditions of validity, it is clear that the global-validity conditions cannot be formulated in this approach in principle, since the perturbation parameter is not a global, basic one. The question of local validity depends on the nature of the series which can be convergent or, alternatively, merely asymptotic (see also section 4.1). In the latter case it is only possible to assert that the approximation is good for α smaller than some threshold value $\alpha_0 < 1$, but the exact value of α_0 remains unknown since the asymptotic series are merely formal ones. [There are examples (see e.g. [74]) where the threshold value of the asymptotic parameter is really very small in comparison to 1.] In contrast, in the case of *convergent* series (and provided the coefficients are of the order of magnitude 1), one has a geometric-series estimate for the remainder of the series in question, and the condition that the leading-order truncation yields a good approximation (i.e. one with the relative error being much smaller than unity) is just $\alpha \ll 1$ (where \ll is used in the *order-of-magnitude* sense--i.e. $\alpha \ll 1$ means $10^{-1/2} < \alpha < 10^{1/2}$ --as opposite to the *asymptotic-order* sense; see [74] for a discussion of the important difference between the two--which is frequently ignored in applied sciences. Following [74], we will use the notation O to denote the expression "...is of an asymptotic order of...", while using O_M for "...is of the order of magnitude of..."). The local condition $\alpha \ll 1$ is the best that one can obtain from such a longwave (LW) approach.

Also, a number of evolution equations have been derived with the LW approach for non-isothermal films (see e.g. [13, 78]); we do not consider those any further, confining ourselves here to the isothermic case.

Single-parameter approach

If there are basic parameters of a *nonzero* order of magnitude, such as $\sigma = O_M(\alpha^{-2})$ above, then one can redefine the perturbation parameter to be a *basic* rather than a local one; e.g. $\Lambda \equiv \sigma^{-1/2}$ --as was the case in the work [89] on the vertical planar films and in the paper [85] on a flow down a cylinder. They, in contrast to the long-wave approach described above, do represent the thickness h as a perturbation series similar to those of other unknown fields. Independent variables are as well rescaled with some powers of the perturbation parameter. We call this technique the (global) *single-parameter* (SP) approach. Here (assuming *convergent* series as discussed above and in section 4.1), one can argue that $\Lambda \ll 1$ is the condition of global validity of the theory. In this respect, we believe the SP approach to be a conceptual improvement (over the LW approach). However, it still has the drawback that artificial dependences are imposed on basic parameters--which are intrinsically independent--since one requires each of them be O_M of one of them, viz. of the perturbation parameter (such as the parameter Λ above). These artificial dependencies between the basic parameters must be included as a part of the validity conditions; they make the validity conditions to be unnecessarily restrictive.

The multiparametric perturbation approach removes those restrictions. As a result, it justifies the theory for much wider parametric domains (essentially, of a higher dimensionality). For example, it was noted [18] (see also [51], pages 266-267) that the validity conditions implied by an SP theory of a film rupture by the van der Waals forces are flagrantly violated by the parameters of realistic films. However, the MP reformulation [32] of the theory led to a much less restrictive domain of validity in the space of parameters, which easily embraced the realistic films. Thus, in this case the MP approach was *necessary* to justify the applicability of the EE under realistic conditions of possible physical experiments.

The desirability of a more rigorous treatment than the above heuristic perturbation approaches are briefly discussed below in section 4.1.

Some results

- [*Three-dimensional inclined-film flow*](#)
 - [General evolution equation; impossibility of a single-equation description of large-amplitude regimes for large times](#)
 - [Evolution equation for small-amplitude regimes](#)
 - [Numerical studies of evolution equation](#)
 - [Unusual patterns on strange attractors for strongly dispersive falling films](#)
 - [Transient patterns: Qualitative agreement of simulations with experiments](#)
 - [*Flow down a vertical fiber*](#)
 - [Large-amplitude regimes](#)
 - [Small-amplitude evolution equation](#)
 - [*Small-amplitude waves in core-annular flows*](#)
 - [*Vertical and horizontal core-annular flows with large-amplitude waves*](#)
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Three-dimensional inclined-film flow

The general 3-D flow can be considered quite similarly to the 2-D flow treatment in section 2.1. In addition, one can allow the plane to be inclined to the horizontal through an *arbitrary* angle θ (between 0 and $\pi/2$); the vertical film of section 2.1 is just a particular case, with $\theta = \pi/2$. This most general case was studied recently in [48] where the reader can find more details. Here, we briefly discuss some of the results.

- [General evolution equation; impossibility of a single-equation description of large-amplitude regimes for large times](#)
- [Evolution equation for small-amplitude regimes](#)
- [Numerical studies of evolution equation](#)
- [Unusual patterns on strange attractors for strongly dispersive falling films](#)
- [Transient patterns: Qualitative agreement of simulations with experiments](#)

General evolution equation; impossibility of a single-equation description of large-amplitude regimes for large times

For $\theta < \pi/2$, there is a component of gravity perpendicular to the plane. This results in a nonzero pressure in the flat-film solution. Accordingly, the locally Nusselt pressure p_0 is introduced in addition to the locally Nusselt components of velocity, the streamwise one w_0 and the cross-film one u_0 . These are given by the same expressions as above (see section 2.1), $w_0 = 2hx - x^2$ and $u_0 = -x^2 h_x$ [now, however, the thickness may depend on the spanwise coordinate y as well: $h=h(y,z,t)$]; similarly, one defines $p_0 = 2(h - x) \cot \theta / R$. Here, the Reynolds number R is defined with the generalized interfacial Nusselt velocity $\bar{U} = \bar{g} \bar{h}_0^2 \sin \theta / (2\nu)$ [above, we denoted \bar{U}_v the value of \bar{U} for the vertical case, $\sin \theta = 1$]; also, \bar{U} (along with \bar{h}_0 and ρ) is used here to nondimensionalize the formulation of the problem equations, the same way as \bar{U}_v was used for the vertical case above.

The new dependent variables are thus u_1, v, w_1 , and $p_1 [\equiv p - p_0]$.

One first solves the x -NS equation, with the normal-stress BC at the free surface, to find p_1 :

$$p_1 = -2(x + h)h_x / R - \sigma \nabla^2 h$$

(here $\nabla^2 \equiv \partial^2 / \partial y^2 + \partial^2 / \partial z^2$). This pressure clearly contains a *viscous* contribution in addition to the usual surface-tension part (and the reference pressure p_0 is of a purely hydrostatic origin). Next, the equations for v, w_1 , and u_1 are solved, in that order. We do not write these results here; they can be found in [48]. Finally, by substituting the velocities (taken at $x=h$) into the kinematic equation, one arrives at the EE which is the generalization of (29) above:

$$\begin{aligned} h_t - \frac{R}{12}(h^5)_{tz} + 2(h^2 - 1)h_x + R \left[\frac{5}{6}h^4 h_x - \frac{3}{10}h^6 h_x \right]_z \\ - \frac{2}{3} \nabla \cdot [h^3 (\cot \theta - W \nabla^2) \nabla h] + 2h^4 \nabla^2 h_x = 0. \end{aligned} \quad (29)$$

Also, the theory yields the local-validity conditions [a generalization of Eq. (10)]:

$$\max \left[\frac{(1+A)W}{L^3}, \frac{(1+A) \cot \theta}{L}, \frac{(1+A)^4 R}{L}, \frac{(1+A)^2}{L^2} \right] \ll 1, \quad (30)$$

where we have assumed, for simplicity, that the lengthscale in the y direction is *not* much smaller than L , the lengthscale of the solution change in the z direction; such so far has been the case in all experiments. [A *one-dimensional* version of the EE (29)--which also lacked the last, odd-derivative term--appeared before, e.g. in [62]. Also, similar to Eq. (29), we have omitted all the dispersive terms other than the last term on the l.h.s. of Eq. (29).] The solutions of this equation yield a good approximation to the true evolutions of the thickness h as long as the conditions

(30) are satisfied.

Similar to our discussion of Eq. (29), the two equations, (29) and the Benney one, are not necessarily equivalent when the conditions (30) break down, and there are indications that our equation has a chance to avoid the explosive solutions which are known (see e.g. [50, 81, 83]) to mar the Benney equation. Indeed, it is the term with the highest power of h (in fact, h^6) in its coefficient which causes the explosion in the Benney equation; but our h^6 -term enters with the *opposite* sign to the one in the Benney equation. Thus, although neither of the two equations is capable of a *quantitatively* good description of the large-time behavior, the EE (29) might be a better choice to be used as a *qualitative* model for large-amplitude waves. This possibility might make this equation worthwhile of a further investigation (which we have not attempted as yet).

Evolution equation for small-amplitude regimes

For small amplitude-regimes, we have $\eta \equiv h - 1 \ll 1$. Substituting $h = 1 + \eta$ into the general EE (29) and neglecting terms which are relatively small [because of $(\eta \sim A) \ll 1$] yields the following evolution equation governing the small-amplitude waves:

$$\eta_t + 4\eta\eta_x + \frac{2}{3}\delta\eta_{xx} - \frac{2}{3}\cot\theta\eta_{yy} + \frac{2}{3}W\nabla^4\eta + 2\nabla^2\eta_x = 0, \quad (31)$$

where by definition $\delta \equiv (4R/5 - \cot\theta)$. From the linear stability analysis (see below) one can see that to have an instability, which is necessary for any interesting nonlinear behavior, one needs $\delta > 0$. We will always assume this condition to hold. It follows that $R > (5/4)\cot\theta > \cot\theta$. One can use this inequality together with the condition $A \ll 1$ to transform the general conditions (30), which leads to the local-validity conditions for the small-amplitude EE (31):

$$\max(W/L^3, R/L, 1/L^2, A) \ll 1. \quad (32)$$

The analysis of the derivation of the EE (31) shows that the third, destabilizing term has its origins in the inertia terms of the NS equations. The two stabilizing terms, the fourth and the fifth, are due to the hydrostatic and capillary (i.e. surface-tension) parts of the pressure, respectively. Finally, the last, odd-derivative, dispersive term is generated by the viscous part of the pressure. Such a term also appeared in the EE which was obtained by Topper and Kawahara [93] who assumed the plane to be nearly vertical (as a result, the hydrostatic term was absent in that equation): They used the small angle of the plane with the vertical as their (single) perturbation parameter. Our derivation shows that this assumption is unnecessary. One can see that for an *arbitrary* inclination θ (and any value of W), provided R is close to $R_c \equiv 5\cot\theta/4$ (so that δ is sufficiently small), the equation (31) can be good, with its dispersive term being not small. At the same time the hydrostatic term can be large (but one needs a sufficiently large spanwise lengthscale Y for this, viz. $Y \gg Z$). If the dispersive term is omitted in the equation of Topper and Kawahara, it becomes the one obtained by Nepomnyashchy [77]. The one-dimensional version of the latter is just the Kuramoto-Sivashinsky equation [64, 87]. Thus, all of these equations--as well as the Zakharov-Kuznetsov equation [98] (see also [65])--can be obtained as certain limiting cases of the EE (31).

Because the EE (31) is dissipative, the system evolves towards an attractor. Thus, it essentially forgets the initial conditions. On the attractor, there cannot be any systematic change in time (although fluctuations around the constant averages may have arbitrarily large amplitudes). Consequently (as was first argued in [6, 37]), the destabilizing inertia term and the stabilizing, capillary one should be of the same order of magnitude. Hence, an estimate of the (dimensionless) characteristic lengthscale *at large times*, L_a , follows:

$$L_a = (W/\delta)^{1/2}. \quad (33)$$

Similarly, the balance between the nonlinear "convective" term and either the dispersive term or the capillary one (whichever is larger) determines the asymptotic magnitude of the characteristic amplitude:

$A_a = \max(W/L_a^3, 1/L_a^2)$. With these estimates, one transforms the condition (16) to the form

$\max(W/L_a^3, R/L_a, 1/L_a^2) \ll 1$. One can finally write this as $\max(R/L_a, L_a^{-2}) \ll 1$ by noting that $W/L_a^3 = \delta/L_a$ and $R = (5/4)(\delta + \cot \theta) > \delta$. Thus the global-validity conditions can be written as Eq. (17) again, but with the generalized definition of L_a given by Eq. (33).

The transformation of Eq. (31) to a "canonical" form is accomplished by rescaling $\eta = N\tilde{\eta}$, $z = L_a\tilde{z}$, $y = L_a\tilde{y}$, and $t = T_a\tilde{t}$, with $N = W/(6L_a^3)$ and $T_a = 3W/2\delta^2$. The resulting small-amplitude evolution equation (with the tildes in the notations of variables omitted) is

$$\eta_t + \eta\eta_z + \eta_{zz} - \kappa\eta_{yy} + \nabla^4\eta + \lambda\nabla^2\eta_z = 0. \quad (34)$$

(with definitions $\kappa \equiv \cot \theta/\delta$ and $\lambda \equiv 3/\sqrt{W\delta}$). This is the generalization of Eq. (18).

Looking at the linear stability properties of Eq. (34), one substitutes the normal mode $\eta \propto \exp(st - i\omega t) \exp i(jy + kz)$ into the linearized version of Eq. (34). This yields the growth rate:

$s = -\kappa(j)^2 - (j)^4 + [1 - 2(j)^2]k^2 - k^4$. From $\partial s/\partial k^2 = 0$, we find the streamwise wavenumber k_{max} corresponding to the maximum growth rate (at fixed j): $k_{max}^2 = 1/2 - j^2$. Hence, the maximum growth rate $s_{max}(\kappa, j)$ is

$$s_{max} = 1/4 - j^2(1 + \kappa). \quad (35)$$

One can see that, for every fixed j , this s_{max} is a (linearly) decreasing function of κ . The results of the linear stability theory are useful for understanding certain results of numerical simulations (see [47, 48]) of EE (34).

The same problem has been treated earlier by Krishna and Lin [63] with the LW approach (see also [84]). They derived a Benney-type equation in the series form (28). They had cited explicitly about one hundred terms of that equation, among which all the terms of Eq. (34) can be found. However, as the MP derivation establishes, only a small number of those terms are essential and should be retained in the equation. All the other terms can be neglected under conditions (32) or (17), i.e. when the equation can yield a good approximation.

Numerical studies of evolution equation

We [48] have numerically simulated the dissipative-dispersive equation (34) with periodic boundary condition. Below, we discuss some preliminary results. It seems plausible that the results will be insensitive to the exact form of the boundary conditions and the size of the integration domain provided the dimensions of the domain are sufficiently large; namely it should include many "elementary structures" (we note that similar simulations give "surprisingly good results" [44] for certain problems of the boundary-layer wave transitions). Accordingly, we solved Eq. (34) on *extended* spatial intervals, $0 \leq y \leq 2\pi p$ and $0 \leq z \leq 2\pi q$, with $p \gg 1$ and $q \gg 1$.

We used spatial grids of up to 256×256 nodes and employed the Fourier pseudospectral method for spatial derivatives, with appropriate dealiasing. Time marching was done (in the Fourier space) with a third-order Adams-Bashforth and/or Runge-Kutta methods. The results were verified by refining the space grids and time steps; by observing the volume conservation, $\int \eta dy dz = 0$; etc.

Some of our initial conditions were motivated by the experiments [72]: these initial conditions modeled the *inlet* conditions of their experiments [see Eq. (36) below].

Below, we discuss some results of our numerical simulations, such as (i) the effect of dispersion on the large-time behavior of the film surface near the attractor and (ii) the dependence of the transient states on the wavenumber of the initial "forcing", as compared to the experiments of Liu *et al.* [72].

Unusual patterns on strange attractors for strongly dispersive falling films

In this section, we consider the most unexpected result of our numerical simulations. Namely, during our studies of the *strongly dispersive* cases of Eq. (34), $\lambda \gg 1$, we found highly nontrivial patterns (of the film surface) persisting for all large times. The system evolves toward these self-organized states starting from the small-amplitude (3-D) *white-noise* surface (whereas with the same, random initial conditions, looking at the large-time states of the surface in the case of *small* dispersion, we see no ordered patterns). This contrasts the spatial patterns studied up to now in fluid-dynamical experiments--as well as in solid state physics, nonlinear optics, chemistry, and biology--which invariably were almost periodic, at least locally (see e.g. [22]). Our results indicate that patterns of quite a different nature can exist on attractors of driven dissipative-dispersive systems. These patterns [48, 47] consist of two *subpatterns* of localized soliton-like deviations of the surface. One of these subpatterns is a V-shaped array of larger-amplitude *bulges* on the film surface. Those move in a "sea" of smaller-amplitude *bumps* which constitute the second subpattern. Each of the two subpatterns spans the entire domain of the flow and moves as a whole along the z coordinate axis. However, the velocity of the bulge formation is different from that of the bump subpattern, so one subpattern "percolates" through the other.

Figures 1 through Fig. 3 illustrate these points. A snapshot of the film surface at $t=3200$, for a vertical film (i.e., $\kappa = 0$) and with $\lambda = 50$ (and $p=q=16$) appears in Fig. 1. (Note that in Fig. 1 the amplitudes of the structures--which in reality are all small-slope--are exaggerated, due to a different scale of the x -axis.) The evolution of "energy" $E \propto \int \eta^2 dy dz$ shown in Fig. 2 demonstrates the time $t=3200$ the film waves have approached their asymptotic state in which there is no further systematic change (although chaotic undulations of the energy are evident).

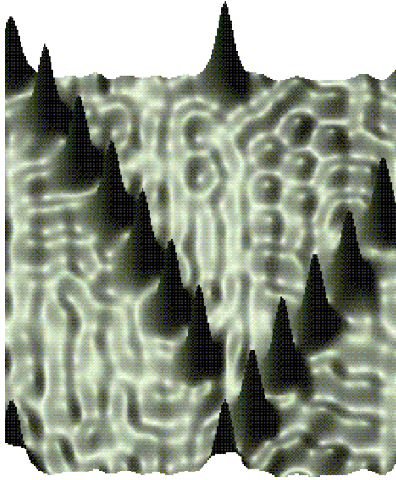


Figure 1: Snapshot of the pattern on an attractor of Eq. (34) with $\kappa = 0$ (i.e. the scaled surface of a film flowing down a vertical plate, here--down the page, with illumination from the top left), $\lambda = 50$, and periodic boundary conditions on $0 < y, z \leq 32\pi$: a view in an oblique direction, for $t=3200$.

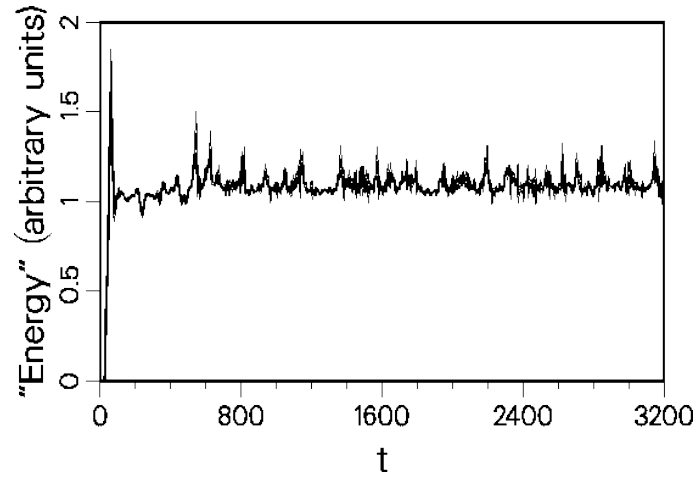


Figure 2: Evolution of the surface deviation “energy” $\int \eta^2 dy dz$ from an initial small-amplitude “white-noise” surface to an attractor of Eq. (34). The snapshots of Figs. 1 and 3 were taken near the end of this run.

One can see the above-mentioned two-stream nature of the pattern in Figures 3(a) and 3(b). The V-shaped formation consisting of 13 large bulges (see also Fig. 1) moves down between the two snapshots shown [the earlier-time one in Fig. 3(a) and a later-time snapshot in Fig. 3(b)]. The small-amplitude background subpattern moves uniformly as well, but in the *opposite* direction (in the moving reference frame of the observer). One can see also that the bump subpattern slowly changes with time. (We have also made a computer animation in which one can clearly see these motions of the two subpatterns.) It had transpired that one component of this dynamical, spatiotemporal pattern, viz. the bulges, has already been observed by the authors of Ref. [92], who performed the simulations of the vertical-film equation with $\lambda = 25$ for $p = q = 64/(2\pi)$. (In fact, they simulated a dissipative-dispersive equation [93] in which the κ term of Eq. (34) was missing, which happened because of a certain--unnecessary, as we discussed above--assumption on which their derivation was based.) However, as far as we know, the authors of [92] used only contour plots as their graphics tools, and overlooked the bump subpattern. Thus, the entire complex, dynamical character of the two-phase pattern remained undiscovered.

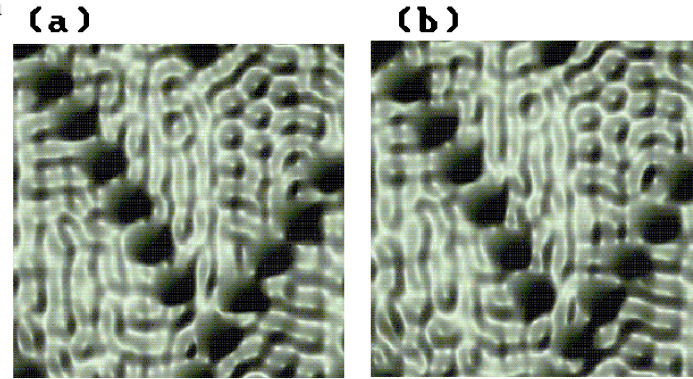


Figure 3: (a) Cross-stream view of the pattern shown in Fig. 1. (b) Same as in (a) but after a time interval $\Delta t = 0.25$. Note that the vertical motion of the V-shaped formation of bulges from (a) to (b) is in the opposite direction to that of the bump pattern.

Figs. 2 and 3 confirm the estimates of characteristic quantities obtained with the term-balancing considerations (see [48, 47] for more on this point).

For the 1-D version of Eq. (34), a perturbation theory of weak interactions (e.g. [9, 8, 42, 27, 56]) of the “pulses” (see [55]) was developed. It is based on soliton solutions of the ODE describing the pulse in a certain reference frame. The soliton, or at least its small outer “tails”--which are responsible for the weak interactions--are found analytically. In contrast, no analytic solution is available for a solitary 3-D bulge. Therefore, the prospects for an interaction theory of the 3-D bulges appear to be uncertain at best.

One observes that the bumps incessantly collide with the bulges. These interactions are seen to be (almost) reversible, like interactions of KdV pulses (e.g. [97]). This contrasts with the irreversible coalescences of 2-D pulses discovered in [38] and [59] for highly nonlinear dissipative equations.

It has been implicitly assumed (see section 4.1) in the derivation of the global-validity conditions for the EE (34) that the solutions of that equation remain bounded. Our simulations justify that assumption. Also, the amplitude of the solutions turns out to be of the same order of magnitude as its estimate obtained by the pairwise balance of terms in the evolution equation.

These simulations also confirm that the large-time behavior of the solutions of (34) is essentially insensitive to the initial conditions: Every solution evolves toward an attractor (whose nature is solely determined by the *basic* parameters of the NS problem).

Similar to Eq. (34), we [29, 36] have derived an evolution equation for a film which flows down a vertical *cylinder* (see Eq. (46) in section 3.2.2). The only essential difference between the two EE is the opposite sign of the κ term.

However, this term disappears in the case of a vertical *planar* film, as well as in the limit of an infinitely large radius of the *cylinder*. According to our simulations, the corresponding Φ^{-2} -term of the annular-film equation (in which the variable ϕ is changed to $y = b\phi$, so that $0 < y < 2\pi b \equiv 2\pi p$) is sufficiently small even for p as small as $p=5$, so

that the results essentially coincide with those of the $\kappa = 0$ version of Eq. (34). So, physical experiments with films flowing down vertical *cylinders* can verify the theory which leads to the planar-film EE (34). However, the cylinder should be sufficiently long so that the waves propagating from the inlet could have enough time to approach the attractor. It is not difficult to find that with $h_0 \sim 1$ mm, the radius ~ 1 cm, and with the restricting conditions (17), in

order for the cylinder to be not too long, the liquid should be sufficiently viscous, in fact several hundred times as viscous as water. This can be easily achieved with e.g. a glycerin-water solution. [It is interesting that one can see a straight row of bulges in the photograph of a film flowing down a cylinder in Fig. 2 of Ref. [12]; however, the R_1 -condition of validity (17) was not strictly satisfied there.]

Transient patterns: Qualitative agreement of simulations with experiments

As was mentioned above, the large-time behavior of the film is insensitive to the initial conditions. In contrast, the *transient* states *do* “remember” the initial condition: the film surface exhibits a variety of patterns as one varies the initial conditions. We studied the transient states primarily in connection with the recent experiments of Liu *et al.* [72], in which the flow (down an inclined plane) was perturbed at its inlet with sinusoidal pressure variations having a fixed frequency f . In some of their experiments, in addition to this single-frequency “forcing”, they used a secondary forcing at the subharmonic frequency $f/2$, with an amplitude considerably smaller than that of the primary forcing. (The objective for the secondary forcing was to enhance possible broad-band subharmonic resonances). We modeled this inlet (possibly, two-frequency) forcing by the initial condition

$$\begin{aligned} \eta(y, z; t = 0) = & A_f \cos(n_f \tilde{z}) + A_h [\cos(2n_f \tilde{z} - \phi_1) + \cos(3n_f \tilde{z} - \phi_2)] \\ & + A_h [\cos(n_f \tilde{z} - \phi_3) + \cos(2n_f \tilde{z} - \phi_4) + \cos(3n_f \tilde{z} - \phi_5)] \cos \tilde{y} \\ & + A_{sub} [\cos(0.5n_f \tilde{z} - \phi_6) + \cos(0.5n_f \tilde{z} - \phi_7) \cos \tilde{y}] \\ & + \text{small white noise,} \end{aligned} \quad (36)$$

(where $\tilde{y} \equiv y/p$, $\tilde{z} \equiv z/q$, and ϕ_i are independently generated random phases), with $A_h \ll A_f$ and $A_{sub} \ll A_f$. (The nonzero A_h takes into account the fact that in practice the monochromaticity of forcing is always imperfect, as well as its axial symmetry.)

For this series of simulations, we have used in most cases the values $\kappa = 0.56$ and $\lambda = 0.06$ which are fairly typical of the experiments of Ref. [72]. However, it turns out that the values of parameters in those experiments lie somewhat outside the domain of validity of Eq. (34). [Since we are dealing with transient states here, one should check the local-validity conditions (32). One finds that the parameter $R_1 (= R/L)$ can be as large as 2 or 3--while it should be much less than unity for our equation to be *quantitatively* good.] Nevertheless, as we will see below, the patterns observed in the simulations turn out to be quite similar to their counterparts in the physical experiments [72].

When the wavenumber k of the primary forcing is close to the neutral wavenumber (i.e. one for which the growth rate $s=0$), which is $k=1$ (for the 2-D modes, $j=1$), and in the presence of the secondary forcing at the subharmonic frequency [i.e. $A_{sub} \neq 0$ in Eq. (36)], we observe in our simulations that the oblique subharmonic interacts very strongly with the fundamental. Fig. 4 shows this for an idealized initial condition,

$$\eta(y, z; t = 0) = A_f \cos(n_f \tilde{z}) + A_{sub} \cos(0.5n_f \tilde{z}) \cos \tilde{y} \quad (37)$$

with $A_f = 1.2$, $A_{sub} = 0.05$, and $n_f = 15$ (while $q=16$). Figure 4(a) shows the strong interaction with the exchange of energy between the fundamental and the (detuned) oblique subharmonics. Plotted there are the “energies” $E_{m,n}$ of the Fourier modes, defined as

$$E_{m,n} = |a_{mn}|^2$$

where a_{mn} is the Fourier coefficient:

$$\eta = \sum a_{mn} e^{i(my/p + nz/q)}.$$

The surface exhibits checkerboard patterns such as the one in Fig. 4(b). One can see that it resembles the checkerboard pattern of "cat eyes" in Fig. 12 of Ref. [72]. However, if one employs the full initial forcing (36), then other modes become significant, as is evident in Fig. 5(a), and while the period doubling is obvious there, only an imperfect checkerboard pattern is observed. (This was further explained in [48] with a simple analysis of the spatial structure of the principal modes.) However, the property of period-doubling remains very robust when the forcing is sufficiently close to the neutral wavenumber. But as the forcing wavenumber decreases away from the neutral, the intensity of the interaction between the fundamental and the subharmonic dies out. Qualitatively similar observations were reported in the experiments by Liu *et al.* [72].

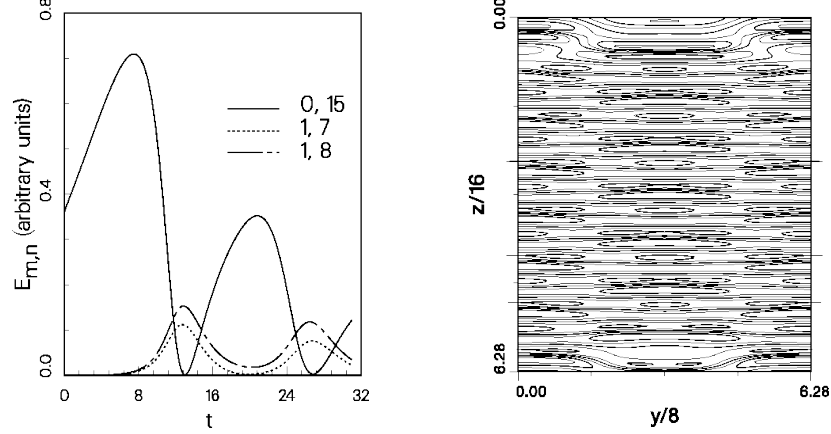


Figure 4: Subharmonic interaction of Fourier modes in a run with the initial condition (37) whose frequency of primary forcing is close to the neutral one ($q = 16, n_f = 15$). (a) Evolution of energies of the principal Fourier modes (the numbers shown in the legend next to each line are the corresponding values of m and n , in this order). Note that since n_f is an odd number, there are two "almost-subharmonic" modes, one with $n=7$ and the other with $n=8$. (b) Checkerboard pattern of the film surface for $t=21$ (cf Fig. 12 of [72]). The interval between two neighboring isothickness contours is 0.6.

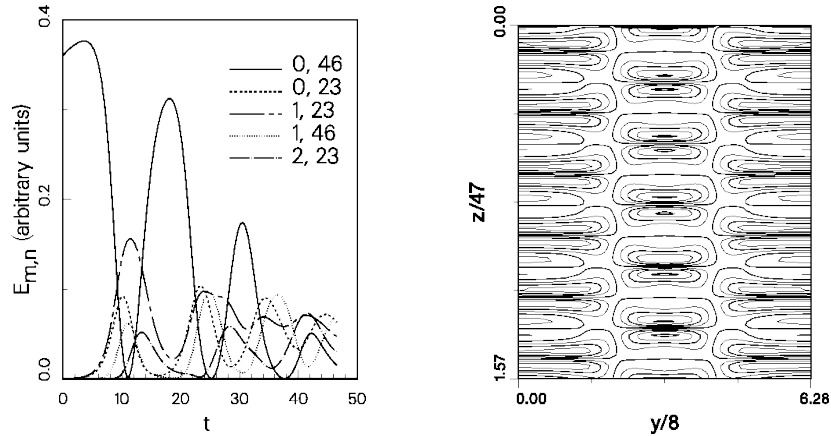


Figure 5: Similar to Fig. 4 but for a run starting from the *general* initial condition (36), with $q=47$ and $n_f = 46$. Note that the checkerboard-like pattern in (b) is not perfect, but the streamwise period-doubling is clear ($t=14.6$; only one quarter of the streamwise extent of the periodicity domain is shown; the interval between two neighboring isothickness contours is 0.6).

With one-frequency initial forcing, there is a certain stability window: When the forcing wavenumber k is between 0.77 and 0.84 and $\kappa \geq 1.55$, the amplitude growth saturates and the final state is a stationary propagating 2-D wave. These

stationary waves consist mainly of the linearly unstable fundamental and a few of its stable overtones with smaller amplitudes. Such stationary waves were constructed by a number of researchers (e.g. [24, 49, 75, 94, 95]) for different film evolution equations; however, it appears that most of the huge number of such stationary solutions are significantly unstable; only a small number of them are observed in experiments--and only when artificial forcing is present. In our case, the stationary waves are well approximated as two-mode equilibria consisting of the unstable fundamental and one stable overtone. Assuming for simplicity that dispersion is negligible, we approximate η as the Fourier series truncated at the N th member,

$$\eta(z, t) = \sum_{n=1}^N A_n(t) e^{inkz} + \text{c.c.}, \quad (38)$$

and obtain from Eq. (34), for $N=2$, the Galerkin system $dA_1/dt = s_1 A_1 - ikA_1^* A_2$ and $dA_2/dt = s_2 A_2 - ikA_1 A_1$, where $s_n = n^2 k^2 (1 - n^2 k^2)$. Hence, for a stationary state ($\partial A_n / \partial t = 0$), and assuming A_1 to be real, we find

$$A_1 = \sqrt{s_1 |s_2| / k}; \quad A_2 = -i s_1 / k.$$

Similarly, a three mode equilibrium including the third harmonic $A_3 \neq 0$, can be found analytically (see [48]). To analyze the linear stability of such equilibria, one considers a normal mode of the Bloch type (introduced a long time ago for the stationary wavefunctions of an electron in the periodic field of the crystal lattice; see e.g. [66]),

$$H \sim e^{Pt + ik(Jy + Kz)} \sum_{m=-l}^l B_m e^{imkz}. \quad (39)$$

After substituting $\eta = \eta_0 + H$ into Eq. (34) and linearizing it, one obtains an eigenvalue problem for P . [Such

“Floquet” analysis was applied before to film waves (e.g. in [49, 17]), and was used earlier in other hydrodynamic problems, see e.g. [44, 57]. We (see e.g. [33, 39, 88, 99]) believe that in this context it is more appropriate to mention Bloch’s rather than Floquet’s name.] Such an analysis [48] confirms the stability to all 3-D disturbances for the values of the parameters indicated above. For the forcing frequencies slightly outside of the stability window, it yields growth rates which lead to valid estimates of the lifetimes of the quasi-stationary states. However, often times such a linear stability analysis fails to predict what patterns will emerge, because the nonlinearity quickly makes dominant other modes than those which govern the linear stage of instability. Thus, the usefulness of the linear theory is quite limited here.

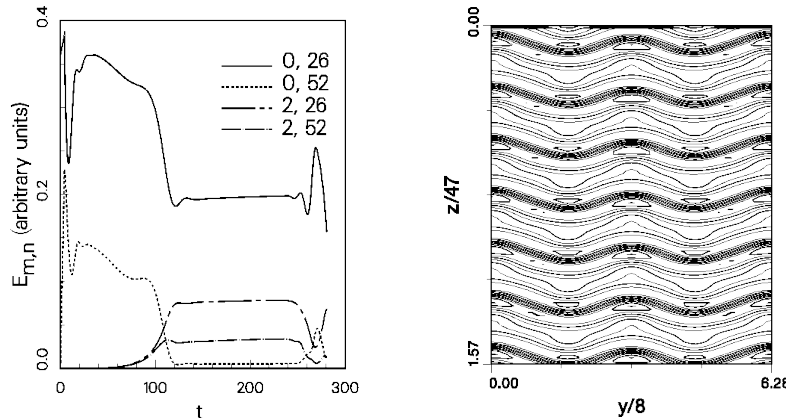


Figure 6: 3-D "synchronous instability" of initially (almost) 2-D waves, Eq. (36) with $A_{sub} = 0$, at lower forcing frequencies ($q = 47, n_f = 26$). (a) Evolution of energies of four principal Fourier modes. Note the quasistationary-wave state beginning at some time after $t=120$. (b) The surface pattern at $t=164$ ($\kappa = 0.18, \lambda = 0.05$: cf Fig. 1(a) of [72]). The interval between two neighboring isothickness contours is 0.4.

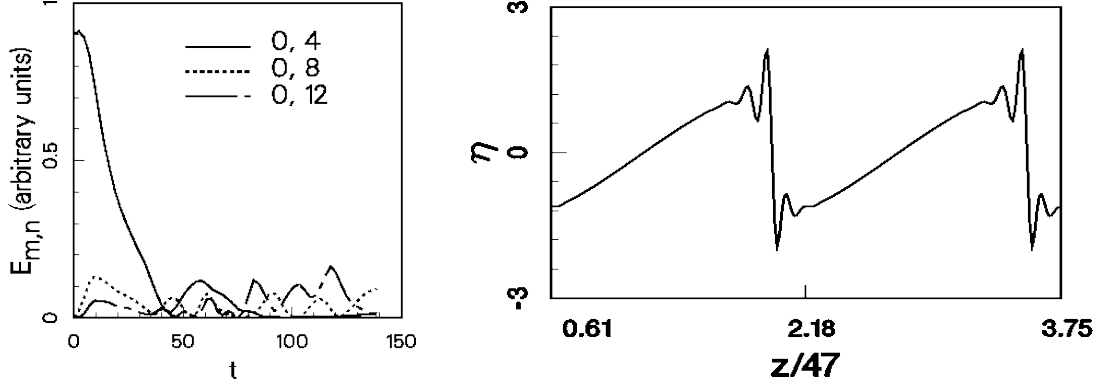


Figure 7: "Solitary wave" pattern in a run with a low-frequency forcing (the initial condition (36), with $q=47$ and $n_f = 4; \kappa = 0.56, \lambda = 0.057$). (a) Evolution of energies of some principal Fourier modes (the numbers shown in the legend next to each line are the corresponding values of m and n , in that order). (b) The wave profile at $t=19.3$. Only a part of the train of four solitary waves is shown (cf Fig. 2(b) of [72]).

In Ref. [48], the reader can find more results (some of them presented in graphics form) of the analysis and numerical simulations of the stationary waves and their stability. In addition, there are results of simulations of "natural" waves, i.e. those which develop from random initial conditions, corresponding to experiments without any forcing.

When the forcing wavenumbers fall below the stability window, we observe evolution which resembles the "synchronous instability" of experiments [72]. Figure 6(a) comes from a run with the $\kappa = 0.18$ and $\lambda = 0.50$, the values calculated with the parameters of the corresponding physical experiment. Evidently, the oblique modes with $m=2$ become important; as a result, a *quasi-stationary* 3-D state develops. The resulting film surface pattern is shown in Fig. 6(b). It should be compared to the experimental pattern found in Fig. 18 of [72]. Some further discussion of the synchronous patterns is found in Ref. [47] (They have been also observed [49] in simulations of the Benney equation on nearly-minimal domains of periodicity.)

When lowering the wavenumbers of the one-frequency forcing even further, we find that the film surface develops *solitary wave*-like transient patterns (see Fig. 7). To a certain degree, these patterns resemble those seen in Fig. 2 of Ref. [72] obtained in low-frequency forcing experiments [however, those experimental solitary waves are seen to have large amplitudes which do not quite satisfy the condition $A \ll 1$, which further lowers the expectations of agreement with simulations of Eq. (34)].

One concludes that these results of simulations of Eq. (34) *qualitatively agree* with the experimental findings of [72] regarding the observed types of transient patterns on inclined films as well as the correspondence between the pattern types and the ranges of forcing frequency.

Flow down a vertical fiber

- [Large-amplitude regimes](#)
- [Small-amplitude evolution equation](#)

Large-amplitude regimes

A theory (of such film flows down vertical cylinders) hinged on a single evolution equations was constructed in [34]. It was motivated by the experiments [82] on flows down vertical fibers. The less-formal version of the MP approach was used to derive the evolution equation for the 2-D waves,

$$h_t + 2h^2 h_z + S \left[h^3 (h_z + h_{zzz}) \right]_z = 0. \quad (40)$$

Here the modified Weber number

$$S = 2\bar{\sigma}\bar{h}_0 / (3\bar{\rho}\bar{g}\bar{b}^3) \quad (41)$$

where \bar{b} is the radius of the cylinder and the other notations have the same meaning as in section 2.1. The nondimensionalization is based on \bar{b} , $\bar{\rho}$, and the "Nusselt" velocity $= \bar{g}\bar{h}_0^2 / (2\bar{\nu})$ (where $\bar{\nu} = \bar{\mu} / \bar{\rho}$ is the kinematic viscosity); however, h is in units \bar{h}_0 . For $S = O_M(1)$, the following parameters were identified which are required to be small for the (global) validity of the theory: the aspect ratio $\beta \equiv \bar{h}_0 / \bar{b}$ and $R_1 \equiv \bar{g}\bar{h}_0^4 / (2\bar{\nu}^2\bar{b}) = (\bar{U}_v \bar{h}_0 / \bar{\nu})\beta$, the modified Reynolds number (the notations here are different from [34]). The appropriate formal MP series (in powers of β and R_1) were also pointed out, and the expressions for velocities and pressure in terms of the film thickness were given. For $S \ll 1$, the theory involves three independent small parameters: S , $\hat{\beta} = \beta^2 / S$, and $\hat{R}_1 = R_1 / S$.

The EE can be written in terms of the surface deviation $X \equiv h - 1$ (in the reference frame moving with the velocity $2\bar{U}_v$):

$$X_t + 2(2 - X)XX_z + S \left[(1 + X)^3 (X_z + X_{zzz}) \right]_z = 0. \quad (42)$$

Also, one can analyze restrictions on the experimental parameters following from the validity conditions for practically available working liquids [34].

In [59], we reported extensive numerical simulations of the EEs (40) and (42) on extended spatial domains $0 < z < 2\pi q$ (where $7 < q < 41$) with periodic boundary conditions. We used (pseudospectral) numerical methods similar to those of section 3.1.3 including the appropriate ("2/5") dealiasing; details can be found in [59].

When a simulation started from small-amplitude random initial conditions, the large-time behavior depended mainly on the value of the control parameter S (provided the periodicity interval was sufficiently large, i.e. $q \gg 1$). For very small S , the surface evolution has (chaotic) KS character (not shown; see e.g. [19, 20, 46, 80]), which is in accordance with theoretical considerations (see [59]). For larger $S = O_M(1)$, first a train of saturated pulses with $O_M(1)$ amplitudes grows, and the further slow evolution proceeds by various interactions of the pulses. The most remarkable discovery in this respect was the irreversible coalescence of pulses; a pulse can grow to a large amplitude by a cascade

of such coalescences, as in Fig. 8. Earlier, such coalescences were observed in physical experiments [70], but to the best of our knowledge the work [59] was the first observation of pulse coalescences in any numerical simulations of PDEs.

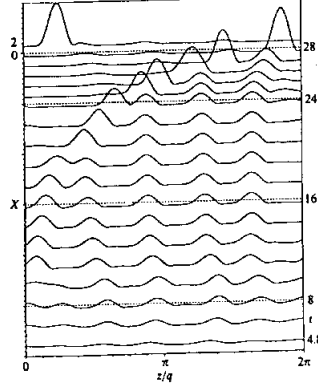


Figure 8: Evolution of the surface deviation governed by Eq. (42) for a large value of the control parameter ($S=3.0$), with a “white-noise” small-amplitude initial condition on an extended interval ($q=8.0$). After (by $t=12$) the initial pulse growth has saturated and a pulse train has formed, a pulse grows by a cascade of coalescences. (From [59].)

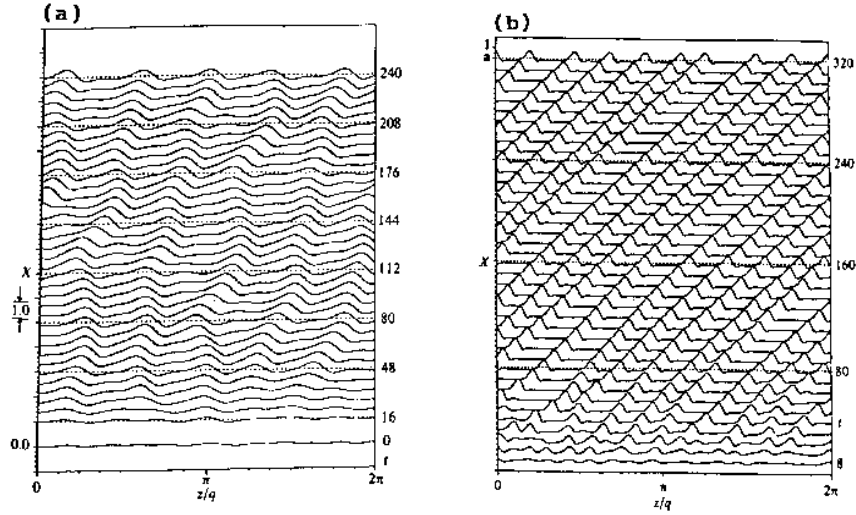


Figure 9: Pulse train evolution for two intermediate (subcritical) values of S in (42). (a) $S=0.4$ ($q=8.0$). (b) $S=0.8$ ($q=20.0$). (From [59].)

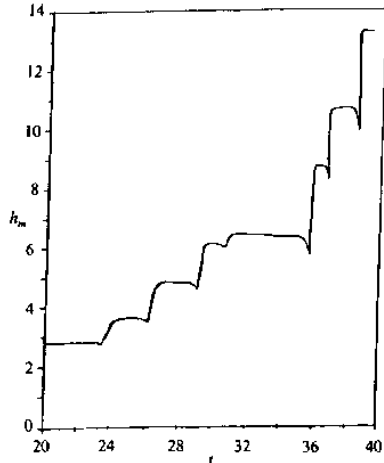


Figure 10: Growth of the amplitude h_m of a pulse in a simulation of Eq. (40) ($S=2.0$; $q=16.0$). (From [59].)

It turned out that the pulse coalescences could happen only for sufficiently large S , greater than certain critical value S_c . (At intermediate values of S , the pulse interactions have the character of particle-like, quasielastic collisions, as in Fig. 9) The simulation results led to $S_c = 1 \pm 0.1$. From Eq. (41), one obtains

$$\bar{h}_{0c} = 1.5 S_c \bar{\rho} \bar{g} \bar{b}^3 / \bar{\sigma}. \quad (43)$$

This is in the form of the Quéré law, who found that the *critical average film thickness*, defined as a thickness value below which no discernible change (in time) of the local film thickness is registered, is proportional to the cube of the fiber radius and does not depend on the film viscosity. In fact, he gave precisely the formula (43) with a single difference: it had a constant he called α in place of $(1.5 S_c)$, and Quéré's experimental value was $\alpha = 1.4 \pm 0.1$.

Thus there is an agreement between our simulations and the experiments [82]. Figure 10 indicates that an isolated mature pulse does not grow between coalescences, contrary to some assertions of Ref. [53] who also simulated Eq. (40) (see [59] for further discussions regarding that work).

At $S \gg 1$, the EE (40) was noted to have the Hammond [43] limit,

$$h_t + S \left[h^3 (h_z + h_{zzz}) \right]_z = 0. \quad (44)$$

This EE describes the film evolution in the absence of both gravity and the average axial flow. However, our results with S even as large as 20 (the largest value we were able to employ) were different from those for the Hammond equation (a notable difference was that for Eq. (40) the pulse coalescences persisted for large values of S). This suggests that the Hammond equation is inapplicable for cases with even very weak (but nonzero) average flow.

The EE (40) can be formally obtained from a Benney-type evolution equation derived in [35] with a LW approach. Before that work, such an equation for the annular films, the analog of the well-known Benney equation for planar films (see also section 3.1.1), was missing in the literature. The key to its derivation in [35] was the assumption of large radius of the cylinder. Previously [68, 2], this problem was considered without that assumption, which led, after very involved calculations, to extremely complicated equations (and the reader should be warned that there are typos in the results of Ref. [68]). One can try to obtain the large-radius EE (40) as a limit of their results, but such a derivation appears to be very complicated and less illuminating than the one in [35] (the authors attempted the former derivation once, but had to give up).

We note that in [35], only the first two orders of the Benney-type equation were given explicitly (we remind the reader that any Benney-type equation has the form of an infinite power series in the small longwave parameter $\alpha = \bar{h}_0 / \bar{l} \ll 1$, as was also discussed in section 2.3.1):

$$h_t + 2h^2 h_z + \alpha \left(\frac{8}{15} R(h^6 h_z)_z + \frac{1}{6} \Xi(h^4)_z + \frac{2}{3} W_2 \nabla [h^3 \nabla (\beta^2 h + \nabla^2 h)] \right) + \dots = 0. \quad (45)$$

[Here (h, t, z) are in units of $(\bar{h}_0, \bar{h}_0 / \bar{U}_v, \bar{l})$, where \bar{l} is the characteristic lengthscale in the axial direction, $\Xi \equiv \bar{l} / \bar{b}$, and $W_2 \equiv W \alpha^2$.] By now, the next-order, α^2 -correction to this equation [such as that found by Nakaya [75] for the planar Benney equation] is also available [73]. One of its members can be shown to correspond to the dispersive term of the small-amplitude EE considered immediately below.

Small-amplitude evolution equation

The first EE for small-amplitude waves on falling annular films was obtained in early 1980s by Shlang and Sivashinsky [85] who used a SP approach. However, by applying the MP machinery, we [36] found that there is a more general equation of that kind; its canonical form is (cf section 3.1.2)

$$\eta_t + \eta\eta_z + \eta_{zz} + \Phi^{-2}\eta_{\phi\phi} + \nabla^4\eta + \Psi\nabla^2\eta_z = 0. \quad (46)$$

Here ϕ is the azimuthal angle, the constant $\Phi \equiv 1 + 8b^2/(5\sigma)$,

$\Psi \equiv 3b/(W\Phi^{1/2})$, $\nabla^2 \equiv \Phi^{-1}\partial^2/\partial\phi^2 + \partial^2/\partial z^2$ and --in terms of the original, based on \bar{h}_0 and \bar{U}_v , units--the new units for $[z, \eta, t]$ are $[L_c \equiv b/\Phi^{1/2}, N_c \equiv W/(6L_c^3), T_c \equiv 3L_c^4/(2W)]$.

The global-validity conditions for this equation are

$$\Lambda_c^2 \equiv 1/L_c^2 \ll 1 \quad \text{and} \quad R_{1c} \equiv W/L_c^3 \ll 1. \quad (47)$$

The equation of Shlang and Sivashinsky [85] is obtained if the last term in Eq. (46) is omitted. Their derivation corresponds to the following choice of powers of the single perturbation parameter (say, $\Lambda, \ll 1$) for the basic parameters:

$$\Lambda \equiv \sigma^{-1/2} \ll 1, \quad b = O_M(\Lambda^{-1}),$$

$$R = O_M(\Lambda^0), \quad L_c = O_M(\Lambda^{-1}), \quad T_c = O_M(\Lambda^{-2}). \quad (48)$$

It follows that $\Phi = O_M(\Lambda^0)$, i.e. the Φ -term should be kept in the EE (46), but the last term can be neglected since $W(= \sigma R/2) = O_M(\Lambda^{-2})$ and therefore Ψ is small: $\Psi = O_M(\Lambda^1), \ll 1$. However (as was discussed in general in section 2.3) these SP conditions are unnecessarily restrictive. There are many other choices of SP powers which satisfy the less restrictive MP conditions (47). For example, changing just two of the SP stipulations (48), to $R \sim \Lambda^1$ and $T_c \sim \Lambda^{-3}$, yields the general dissipative-dispersive equation (46). Another choice, $R \sim \Lambda^{-1/2}$ ("large" R !) and $T_c \sim \Lambda^{-3}$ gives the EE of [85] again (but with the amplitude $A \sim \sqrt{\Lambda}$ instead of their $A \sim \Lambda$).

The presence of *two* terms in the expression $\Phi \equiv 1 + 8b^2/(5\sigma)$ reflects the fact that there are two destabilizing influences, which are (i) inertia [the same as for the planar film, Eq. (34)] and (ii) the transverse-curvature part of the capillary pressure (which does not exist for the planar film). Their balance with the stabilizing longitudinal-curvature part of the capillary pressure determines the longitudinal (i.e. axial) lengthscale. If

$$\sigma \ll b^2,$$

the capillary destabilization is negligible in comparison with the curvature one, and the EE reduces to that of the planar film (34). One can readily find that in this case $\Phi = O_M(b^2/\sigma)$ and $L_c = O_M(\sqrt{\sigma})$; also, the condition of

negligibility of dispersion, $\Psi \ll 1$, can be written (recall the relation $W = R\sigma/2$) as $1/\sqrt{WR} \ll 1$, the same as for the vertical planar film (in this case, the validity conditions (47) reduce to $R/\sqrt{\sigma} \ll 1$). In the opposite case, i.e. when \sqrt{WR} is sufficiently small, so that the dispersive effects cannot be neglected, the validity condition is just $\sigma \gg 1$. From the above considerations, the "cylindricity" of the film can play a role only when b is sufficiently small-- $O_M(\sqrt{\sigma})$ or less. Otherwise, we have essentially the problem of a planar vertical film (with the restriction that the boundary condition in the spanwise direction must be periodic.) Such appears to be the case in all experiments documented in the literature (e.g. [62, 12, 54]) with the exception of [82]. The analysis of the above condition for the practically available liquids shows that for the cylinder curvature to be essential, the cylinder should be a microscopic fiber or the film a very viscous liquid (or both). In this case, $L_c = O_M(b)$. The condition of dispersion negligibility becomes

$$b/W \ll 1.$$

When curvature and dissipation are essential factors, the (single) validity condition is $W/b^3 \ll 1$. When curvature and *dispersion* are both essential, the validity condition can be written in the form $1/b^2 \ll 1$. One can see that in fact in all those four cases (of possible domination outcomes in the competing pairs, inertia versus curvature and dissipation over dispersion)--that is in all cases where the EE (46) (or its simplifications, such as the EE of Shlang and Sivashinsky [85]) is valid--it follows (from the domination conditions and the validity conditions) that

$$b \gg 1.$$

(This has in fact been used to simplify expressions for coefficients of Eq. (46): in some places where $(b+1)$ enters, we changed it to just b .)

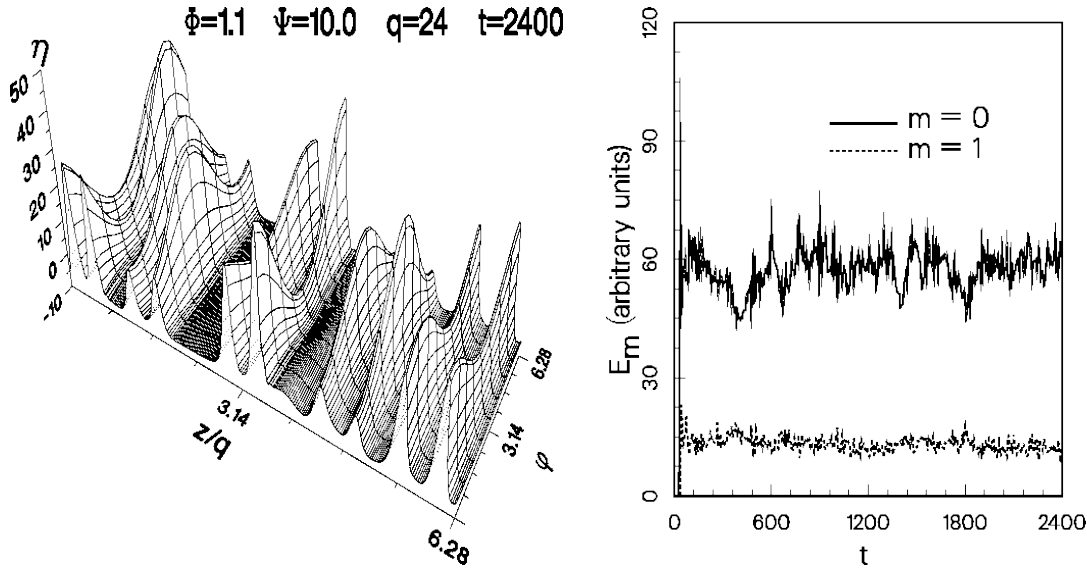


Figure 11: (a) Snapshot of the surface of a film (flowing down a cylinder) which develops from a small-amplitude white-noise initial surface in a simulation of evolution governed by Eq. (46). Due to nonlinearity, azimuthal variations have grown, despite the stability of nonaxisymmetric normal modes of the linear theory. (b) Evolution of modal energies; the snapshot of (a) corresponds to the end of this run.

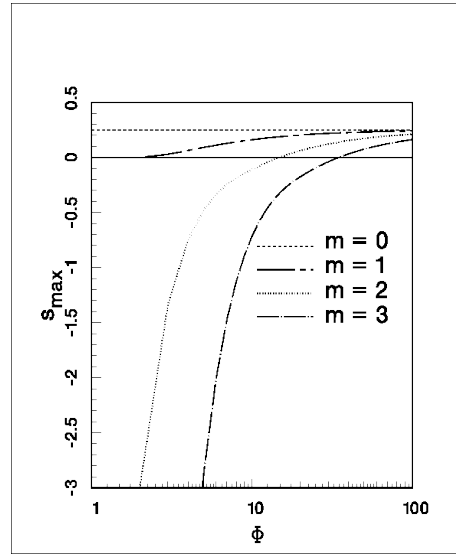


Figure 12: Maximum (over the axial wavenumbers) growth rate of normal modes (with the indicated values of the azimuthal wavenumber m) of linearized Eq. (46), as a function of the “curvature parameter” Φ .

For the equation of [85], that is Eq. (46) without the dispersion term, 3-D simulations on extended domains were done by Deissler *et al.* [23]. It is not surprising that in those simulations the azimuthal dependences developed: for their values of parameters, there are linearly unstable non-axisymmetric modes, as one can easily see from the linear stability analysis. However, only axisymmetric modes are unstable for $\Phi < 2$. Nevertheless, the azimuthal structure appeared

in our simulations for Φ as small as $\Phi = 1.1$ (see Fig. 11). The explanation is that some modes, of the azimuthal wavenumber equal to unity, are only “weakly stable”: their “decay” rate is exactly zero (see Fig. 12), i.e. they are *neutrally* stable (or, one can say as well, neutrally unstable). Therefore, the nonlinear interaction with unstable axisymmetric modes can easily grow these nonaxisymmetric ones.

However, the modes with azimuthal wavenumber higher than unity are strongly stable and hence do not grow. We believe this is the reason that we did not observe any coherent structures--such as the bulges of Fig. 1-- in our strong-dispersion simulations, $\Psi = 10$, for $\Phi < 4$: as was mentioned before, Fourier modes with many azimuthal

wavenumbers (and many axial ones as well) are needed in order to make such a coherent structure.

Small-amplitude waves in core-annular flows

A perfect core-annular flow (CAF) is axially symmetric and unidirectional: Two immiscible fluids fill a circular pipe and flow parallel to its axis. The interface between the *annular fluid* and the *core* one is a perfect cylindrical surface.

However, it is well known that the surface tension at a cylindrical interface is a destabilizing factor (e.g. [25]). The jump in viscosity at the interface can be destabilizing as well. The linear-stability studies of CAFs have a long history (see e.g. [52]). In [51], the numerical studies of D. D. Joseph and his collaborators are presented in detail. They have greatly extended our knowledge of linear stability properties in different domains of the space of basic parameters (see also [90]). They also studied CAFs experimentally, and compared the results with the linear theory. However, their nonlinear Ginzburg-Landau-type theory yields nearly monochromatic waves modulated on large lengthscales. Such waveforms were never observed in experiments--presumably because the parametric domains of validity were extremely narrow.

We [37] have suggested a nonlinear MP theory for core-annular film flows (CAFFs; in which the annular fluid is a film, i.e. its thickness is much smaller than the core radius) which yielded interfacial waveforms in closer resemblance to those observed in experiments (this work is discussed in [51] as well). The paper [37] was, to the best of our knowledge, the first nonlinear study of CAFFs from the first principles of hydrodynamics. (The nonlinear evolution of core-annular configurations with *no basic flow* was considered earlier by Hammond [43].)

In [37], we studied the *horizontal* CAFF, driven by an axial pressure gradient, and such that the effects of gravity are negligible [which is the case when a certain--called Bond's--number (whose definition can be found e.g. in [37]) is small, due to either the closeness of the densities or the small thicknesses of the liquid layers]. We derived (with the less-formal version of the MP approach) an EE for small-amplitude regimes in which the influence of the core disturbances on the film can be neglected. This EE turned out to have the structure of the KS equation. The global-validity conditions were obtained as three inequalities which can be written in the form

$$\frac{h}{b} \ll 1$$

and

$$\frac{\sigma h^3}{\mu b^3} \ll U_G \ll \frac{\nu}{h}$$

(where b is the pipe radius, U_G the velocity of the basic interface, and the rest of the notations are as before; all

quantities here are dimensional, but, for brevity, we have omitted the overbars). This work has demonstrated that the presence of basic flow can lead to saturation of the capillary instability, and thus keep from breaking down the core-annular topological type of the arrangement of the fluids.

Similar KS evolution equations were obtained for small-amplitude cases of two-fluid horizontal flows [6, 86, 45]. In [6], it was shown that the basic plane Couette flow can defy the Rayleigh-Taylor instability, thus keeping as topologically invariable the arrangement in which a film of a *lighter* fluid is at the bottom. (Similar results were obtained for a sheared film of a single fluid destabilized by molecular forces [5]). The other two papers considered a plane Poiseuille flow of two fluids in a channel (see [52], section IV.8, for a review). Recently, Benney-type equations were obtained for two-fluid flow in an inclined channel by Tilley *et al.* [91]. Such equations for large-amplitude cases can only be good as qualitative models, as was discussed above.

A more realistic core-annular problem than in [37], viz. a problem of CAFF with nonequal viscosities, was solved in [31] with the MP approach (later this solution was reformulated with an SP approach by Papageorgiou *et al.* [79] whose numerical simulations of the EE of [31] had confirmed the conclusions of [31].) In that work, we had shown that the core disturbances can make a nonlocal contribution expressed by an integral-operator term in the evolution equation. (It was not written explicitly there--because of space limitations required by the publisher, but it was clearly indicated in [31] how to obtain that integral term ``...in terms of the (confluent hypergeometric) Kummer's function..." [31].)

By using certain properties of the (dispersive-dissipative) integral term, we established several domains in the space of basic parameters for which no break-up occurs, so that the flow keeps its core-annular character--including some cases in which the Reynolds number of the core was *much greater than unity*. (In fact, each such domain is given by the corresponding conditions of global validity of the theory.) When this core Reynolds number is much less than unity, the integral term of the EE, as we pointed out in [31], ``...simplifies and is found in terms of modified Bessel functions..."

The idea of the formal MP approach was first introduced in that work. We also pointed out the importance of the dissipative terms even when they are much smaller than the dispersive terms coming from the nonlocal, integral member of the EE (similar to [55]). The possibility of a linear-mechanism stabilization (by the viscosity stratification) in the case of a more viscous film was demonstrated.

A generalization of this work to the case of rotating CAFF was undertaken in [21].

Vertical and horizontal core-annular flows with large-amplitude waves

A nonlinear evolution equation for 2-D waves in a vertical CAFF driven by gravity and/or vertical pressure gradient were cited in [38]:

$$h_t + (U - Q)h h_z + 2Qh^2 h_z + S \left[h^3 (h_z + h_{zzz}) \right]_z = 0. \quad (49)$$

Here the units of the film thickness h , basic velocity of the interface U , time t , and axial coordinate z are the Nusselt film thickness h_0 , the Nusselt velocity $U_v \equiv gh_0^2/(2\nu)$, the time h_0/U_v , and the tube radius, respectively;

$Q \equiv 1 - \rho_c/\rho$, where the last term is the core-to-film ratio of densities; and S is a modified Weber number $S = \sigma\beta^3/(3\rho\nu U_v)$ where σ is the surface tension (we have omitted for brevity the overbars in notations of the dimensional quantities) and β is the ratio of film thickness to the tube radius.

One [28] can obtain this equation, valid in certain domains of parameters, either directly from the NS problem with a MP approach, or formally from the 3-D Benney-type equation [28]

$$h_t + (U - Q)h h_z + 2Qh^2 h_z + \alpha \left(\frac{8}{15} R(h^6 h_z)_z + \frac{2}{3} W_2 \nabla [h^3 \nabla (\beta^2 h + \nabla^2 h)] \right) + \dots = 0 \quad (50)$$

[here (h, t, z) are in units of $(\bar{h}_0, \bar{h}_0/\bar{U}_v, \bar{l})$, where \bar{l} is the characteristic lengthscale in the axial direction, $\Xi \equiv \bar{l}/b$, and $W_2 \equiv W\alpha^2$) which can be derived [28] analogous to the similar equation [35] for the vertical annular film [see Eq. (45) in section 3.2.1]. In the regimes Eq. (49) describes, the effect of the core disturbances is negligible; additionally, inertia effects are negligible. (This equation is good for CAFFs in vertical *capillaries*, but cannot be expected to describe the "bamboo waves" discovered in "macroscopic" experiments [7]: indeed, as is easy to estimate, the inertia effects are important in those experiments.)

When the pressure gradient is absent, and in the limit of a vanishing core, $\rho_c = 0$, one finds that $U \approx 1$ and $Q=1$, and Eq. (49) reduces to the Eq. (40) of a falling single film. In another limit, $g=0$, Eq. (49) reduces to the EE of the horizontal CAFF which first appeared in [3],

$$h_t + h h_z + S_G \left[h^3 (h_z + h_{zzz}) \right]_z = 0. \quad (51)$$

Here the Weber-type number S_G is defined similar to S in Eq. (49) but with the basic interface velocity U_G instead of U_v . The same change of basic velocities is used in definitions of the units for the basic velocity of the interface U , time t , and axial coordinate z . Note that the structure of this EE differs from that of the falling-film EE (40) by the power of h in the coefficient of h_z (the advective term) only.

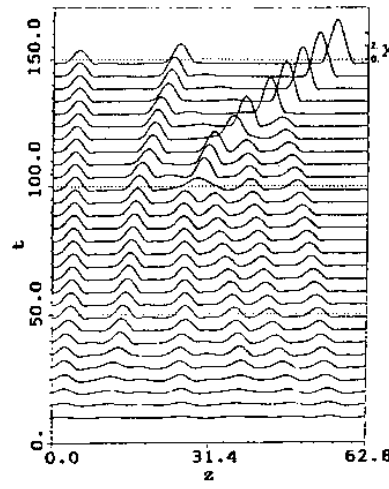


Figure 13: Development of a pulse train and the cascade of pulse coalescences in the horizontal core-annular flow governed by Eq. (51) with $S_G = 0.5$ ($q=10.0$). (From [38].)

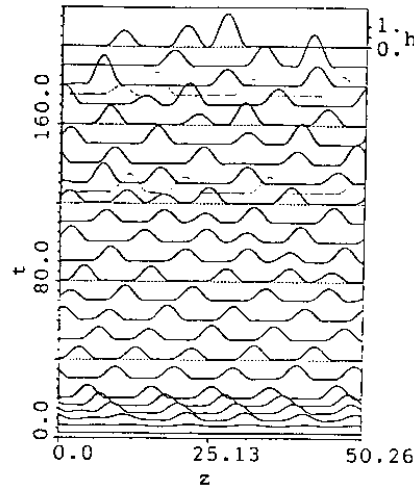


Figure 14: Coalescences of pulses in a vertical core-annular flow governed by Eq. (49) with $U=2.5$ and $Q=-0.5$, for $S=2.0$ ($q=8.0$). (From [38].)

In [38], we simulated EEs (49) and (51) on extended spatial intervals with periodic boundary conditions, similar to the work [59] for the falling-film equation (40). [Also, later the same results appeared in [58] (with some elaborations and along with numerical simulations of some model equations of the horizontal CAFF)]. For the horizontal-CAFF equation (51), we detected some errors in earlier finite-difference short-interval simulations of [3]. In general, as the control parameter increased, we observed the same windows of qualitatively different behaviors as for the EE (40) (see [59]), but with lower boundaries between the neighboring windows. In particular, the coalescences of two pulses one of which is sufficiently larger than the other (see Fig. 13) occurred for S_G as small as the critical value $S_G = 0.15$.

Cascades of such coalescences can result in formation of "giant" collar-like bulges of the film, which eventually may become "lenses" spanning the entire cross-section of the capillary tube. The lens formation can be regarded as a result of a pinch-off of the core, such as those observed in physical experiments [3, 4] for even very small average thicknesses, such as $\beta = 0.04$. The mechanism in Ref. [40] (proposed there for a core-annular configuration without any mean flow), if applied to the CAFF, would predict no pinch-off for $\beta < 0.12$. Our results suggest that coalescences can raise a local mean thickness over the threshold value, allowing the mechanism [40] to pinch-off the core. Thus, the coalescences of pulses explain the formation of lenses in experiments [3, 4], which formerly remained unexplained.

Such coalescences have been observed [38] for general vertical CAFFs as well (see Fig. 14).

Some unresolved questions concerning foundations of the film flow research

- [On foundations of perturbation methods used in film flow studies.](#)
- [On numerical simulations of evolution equations](#)

On foundations of perturbation methods used in film flow studies.

Although the history of the use of *perturbation methods* in nonlinear studies of film flows is at least several decades long, in all cases we know of they have been used in only formal and heuristic ways, quite far from being mathematically rigorous. Every time that a formal power series is obtained to represent a solution, one can ask whether that series is convergent and what is its radius (or radii, in the case of multiple series) of convergence. If it is divergent, the series can be *asymptotic* to the exact solution and thus still useful for approximating it. The worst possibility is that the series is *not* even asymptotic to the true solution; in that case, using it to represent the true solution can be plainly misleading. Even if the series is asymptotic, this asymptoticity can be *uniform* or not. To the best of our knowledge, such questions have never even been mentioned in the literature on film flows. We believe it is important to discuss these questions even if few, if any, of them can be answered at the present time. (For terminology we use and a good introduction to perturbation methods, the reader can consult the book by Murdock [74]; see also [60, 76, 26].)

It is clear that in all papers which employ the perturbative power series, it is implicitly assumed that those are at least asymptotic to the true solutions. Considering the long-wave theory, the perturbation parameter α is based on the *initial* lengthscale (rather than on basic parameters). The global, i.e. uniform in time, validity of (i.e. a good approximation by) the series in α must include their validity at large times, at the attractor--on which the characteristic lengthscale of the solutions may be quite different from their initial lengthscale. The attractor lengthscale is determined by the basic parameters, and the initial parameters are essentially irrelevant to it. Therefore it is clear that the global-validity conditions (see section 2.1) cannot even be formulated in terms of the longwave parameter.

Proceeding to theories based on *basic* perturbation parameters, let us speak for simplicity of a single-parameter perturbation approach, where the series are in powers of a parameter (say) ε . If a theory is merely asymptotic, all one can assert is that the approximation is good for ε less than some ε_0 , and there are examples when the threshold value ε_0 is extremely small (e.g. [74, 67]) so that just $\varepsilon \ll 1$ is not the correct validity condition. Since the value ε_0 is unknown

a priori, one cannot formulate the validity conditions in terms of ε . Analysis of the relevant film-flow literature, however, leads to the conclusion that the validity condition $\varepsilon \ll 1$ (in the sense of the order of magnitude, and not of

the asymptotic order) is implicitly assumed there. That implies convergent series of the type of the geometric series $1 + \varepsilon + \varepsilon^2 + \varepsilon^3 + \dots$, whose sum is $1/(1 - \varepsilon)$ and the error in approximating this sum with the first term of the series, the number 1, is $\varepsilon/(1 - \varepsilon)$, which is $\ll 1$ if $\varepsilon \ll 1$. It is plausible, in our view, that in most cases of film

flows the perturbation theory is regular, in the sense that it includes just such convergent series (if there is a *negative* power of the parameter at the beginning of a series--as in the series for p in Eq. (21)--it is *not* a significant singularity: it can be regularized by defining a new variable to be the series divided by that negative power). Indeed, as was mentioned in the Introduction, the (nonhomogeneous) equations for the higher-order coefficient functions (of the series for the unknowns) have coefficients made of the lower-order CFs. If the latter are $O_M(1)$, the coefficients of the equations in question are $O_M(1)$, and thus the solutions, the next-order coefficient functions, are likely to be $O_M(1)$.

Inductively, one concludes that all the coefficients in the series for the unknowns are of order of magnitude 1, so that the series are of the geometric type.

However, this argument is obviously merely plausible rather than rigorous. As time goes to infinity, the solutions of a

PDE with coefficients of order one can (i) remain bounded; (ii) decay to zero; and (iii) grow to infinity. The argument under consideration ignores the unpleasant possibility (iii). This can be excluded by integrating numerically the systems of different orders for the coefficient functions. So far, this was restricted to just the leading-order simulations in the film flow studies (for some higher-order simulations in investigations of *jets*, see [10]). It would be interesting to explore the simulations for higher orders in future studies of film flows.

The same considerations apply to the (formal) multiparametric perturbation approach. The (global) validity conditions assume convergent (multiple) series of the geometric type. Since such a convergence has not been proved, the method remains heuristic, just as the longwave and the single-parameter approaches. [However, as we argued in the Introduction, the advantage of the MP approach is that it yields the validity conditions (albeit heuristic as well), which are much less restrictive than those following from the SP derivation.]

In the less-formal MP approach demonstrated in the Introduction and used in most of our theories, the validity conditions follow in fact from what is termed "the apparent consistency of the basic simplification procedure" in the excellent book [67]. However, in Chapter 6 of that book, it is demonstrated (albeit for algebraic examples only) that sometimes the presence of the apparent consistency does not imply that the solution of the simplified equations is a good approximation to the exact solution of the original, full problem. Therefore, in principle, a rigorous proof is required that our validity conditions really guarantee the good approximation. Until such a proof is available, the less-formal MP remains only heuristic as well.

Even when the perturbation series are just asymptotic rather than convergent, the MP approach can be preferable since a single MP theory is equivalent to infinitely many SP ones (with different scalings of the basic parameters as powers of the single perturbation parameter). However, the MP theory uses the assumption that if a series is equal to zero, then each coefficient separately must be equal to zero. The proof is well known for the usual asymptotic series containing a single perturbation parameter, but one can see that it cannot be immediately generalized to the case of asymptotic series with many independent asymptotic parameters. Nevertheless, the uniqueness of asymptotic series *can* be proved at least for the case of two perturbation parameters [30].

Finally, the questions touched upon in this section are not specific to the theories hinged on a *single* evolution equation, but are as well relevant to theories based on less drastic simplifications of the NS problem, such as the boundary-layer system of Refs. [17, 16]. Indeed, such theories can only be justified by a perturbation theory using one or more small parameters.

On numerical simulations of evolution equations

Most simulations in the film flow literature use periodic boundary conditions, which make the problem easier. As was mentioned before, it is desirable that the interval of periodicity be sufficiently *large*, containing many elementary structures of a characteristic lengthscale; then one can hope that the result do not depend on the size of the interval, except for the edge effects.

However, in any case, the problem with periodic BC is an initial value problem, whereas in the physical experiments with the convectively unstable [72] films an important phenomenon is the downstream amplification of the *inlet* noise. This can be modeled with different BC [23, 14], corresponding to the spatial (as opposite to temporal) development of instability. Though we do not expect a significant difference between the two kinds of numerical simulations, the spatial one and the temporal one that we have been doing (indeed, one can think of a wave packet which is emitted at the inlet; after that, in its co-moving reference frame, it just develops from that initial condition, as it is advected downstream by the film flow), still it would be interesting to make the spatial 3-D simulations of our evolution equations and compare them with the periodic-BC-simulation results. (One source of a possible difference is the fact that with the periodic conditions, a structure exiting from the downstream end of the interval exits the upstream end, and thus can catch up with another, *slower* moving structure, which originally was *upstream* of the first structure, so that the two would never interact on an unlimited interval; whereas they come to interact on the limited interval of periodicity.) Such 3-D spatial-evolution simulations have never been done before with the more complicated boundary-layer systems, because of computational difficulties (see [14] for their 2-D simulations). However, for a single evolution equation they are quite feasible, as is evidenced by such simulations [23] of the equation of Ref. [85].

Another question one should be aware of concerns long-time simulations of a system evolution to a *strange* attractor. Since there is a sensitive dependence of a trajectory on the attractor to the initial conditions, the computed trajectory close to the attractor can relatively fast wander away from the true trajectory starting from the same initial conditions. This can happen no matter how good the numerical method is, just because of the unavoidable round-off errors. (This is a kind of the *ill-conditioning* problem.) Nevertheless, we believe it can happen that at the same time the computed chaotic trajectory reflects all the important (statistical?) properties of the true trajectory, so that the computed results are “good” for all practical purposes. We have not seen this problem discussed in the literature.

In simulations of small-amplitude EEs of flows down cylinders it was always assumed that the waves are axisymmetric because there are no unstable nonaxisymmetric modes in the linear theory. However as we mentioned above, some first (i.e. with $m=1$) nonaxisymmetric modes can be only weakly, neutrally, stable, so that nonlinearity can easily excite them. This circumstance should be kept in mind for the future studies of the falling annular films.

Summary

For different film flows, we have discussed theories hinged on a single evolution PDE. Such theories can be obtained by the multiparametric perturbation approach. Its advantage over earlier perturbation methods is that along with an evolution equation--and explicit expressions for velocities and pressures in terms of film thickness--it also yields conditions on parameters under which the theory provides a good approximation of the wavy film evolution.

Evolution equations are now available, with known parametric domains of their validity (local and global), for small-amplitude regimes of planar--inclined or vertical--film flows. Such theories, each hinged on a single evolution equation of the film thickness, have also been obtained for waves in the general vertical core-annular film flow, and in its limiting cases--the flow of a free-surface film down a cylinder and the horizontal core-annular flow; in all these cases the evolution equations are available for both the small-amplitude and large-amplitude (but still small-slope) regimes. The large-amplitude regimes of *planar-film* flows cannot be quantitatively approximated by a single evolution equation of the film thickness. The simulations of the Benney-type equations (which equations are now available for the cylindrical cases as well--for the large-radius falling annular film and for the general vertical core-annular flow) should be used as qualitative models only.

The small-amplitude evolution equations are relatively amenable to fully-dimensional numerical simulations on extended spatiotemporal domains. These can lead to insights useful for the studies of even large-amplitude waves. Simulations of the equation for small-amplitude regimes of a strongly dispersive falling planar film exhibit formation of unusual two-stream 3-D patterns at large times, near a strange attractor.

Some large-amplitude, highly-nonlinear equations have been simulated on extended spatiotemporal domains for two-dimensional waves. Here, the most significant finding appears to be the phenomenon of irreversible coalescences of interacting solitary pulses.

There are several agreements of these theories with corresponding experiments: quantitatively with the experiments on flows on vertical fibers [82]--whose parameters did satisfy the conditions of validity of the theory; and qualitatively with the inclined-film experiments [72] whose parameters were somewhat outside of the domain of theory validity. Also, the collision of pulses may qualitatively explain the phenomenon of core pinch-off in the capillary core-annular flow [4] (where the pulses grow beyond the domain of the theory validity).

Many fundamental questions in the study of film flows have not yet been answered. Nevertheless, significant progress is under way, and many interesting insights and findings can be expected in the near future.

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