Wavy film flows down an inclined plane: Perturbation theory and general evolution equation for the film thickness

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Wavy film flow of incompressible Newtonian fluid down an inclined plane is considered. The question is posed as to the parametric conditions under which the description of evolution can be approximately reduced for all time to a single evolution equation for the film thickness. An unconventional perturbation approach yields the most general evolution equation and least restrictive conditions on its validity. The advantages of this equation for analytical and numerical studies of three-dimensional waves in inclined films are pointed out. [S1063-651X(99)10610-X]

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I. INTRODUCTION

Thin liquid layers (''films'') flowing along solid surfaces—such as inclined planes or vertical cylinders—occur in both natural and man-made environments. Industrial applications of film flows started as long ago as the 1800s and have been growing in their scope and importance ever since (see, e.g., Refs. [1,2]).

Accordingly, the studies of film flows, in particular those down an inclined plane, have a considerable history (see, e.g., Refs. [1,3]). However, the dynamics of nonlinear waves (typically present in film flows) is still far from being satisfactorily understood (see Refs. [4,5] for the most recent progress reviews). The nonlinear Navier-Stokes (NS) partial differential equations (PDEs) of this ''Kapitza problem'' couple together several fields (pressure and the components of velocity), each a function of time and three spatial coordinates. Furthermore, the boundary conditions (BCs) of the problem involve a free boundary (the free surface of the film) whose PDE itself is coupled to the NS equations. The full spatially three-dimensional (3D) problem is too hard to simulate even with the most powerful modern computers.

Even simpler, 2D computations of wavy films have been undertaken only under the simplifying assumptions of short spatial intervals (e.g., Refs. [6,7]) and/or time independence (e.g., Ref. [8]). However, 2D flows are frequently unstable to 3D disturbances, and three dimensionality can be important for many inclined films (see, e.g., recent experiments in Ref. [9]).

Therefore, naturally, one looks for more manageable *approximate* descriptions of wavy-film evolution. Such simplified theories are possible in certain domains of the space of parameters (for which, typically, the slopes of the surface waves are small). Of course, the greater simplification has to be paid for by a more limited applicability of the theory. The greatest simplification is achieved when the problem reduces to a single evolution PDE approximating the thickness of the film. [From the numerical-simulation point of view, the most important simplification here is the reduction in the number of independent spatial variables, and even a lower-dimensional *system* of equations would have been not much more difficult than a *single* evolution equation (EE). However, no such system has ever been derived consistently for an inclined single layer film, the subject of our consideration in this paper. The same is true for the *nonlocal*, i.e., integro-differential, film-thickness equations. (The only known example [18] of such a nonlocal EE, as far as film flows are concerned, is an EE for a core-annular flow, and not for an inclined film.) In any case, in this paper, the nonlocal equations are altogether excluded from the consideration; thus, when we say ''the most general EE,'' it should be understood as ''the most general *local* EE,'' etc.] Although less drastic simplifications than a single EE (and therefore having a larger parametric range of validity) are known (see, e.g., Ref. [5]), so far fully dimensional simulations for sufficiently extended spatial domains have been carried out [4,10] only for the theories hinged on a single evolution equation. Such single EE theories of inclined-film flows are the subject of the present paper.

Evolution equations for film thickness have been known since the pioneering work of Benney [11]. The conventional perturbation approach to their derivation (e.g., Refs. [12–15]) used a small (long-wave) parameter, say ϵ. In particular, each of the ''global'' (''internal,'' ''basic'') parameters specifying the problem (the parameters appearing in the dimensionless NS equations and the free-surface BCs) must be ascribed, in such a single-parameter (SP) technique, a certain power of ϵ as its order of magnitude (O_M). Therefore, an artificial dependence is forced on the—intrinsically independent—parameters. This unnecessarily restricts the domain of justified validity of the resulting EE. For example, for the vertical film, there are just two independent parameters: the ''Reynolds number'' R and the ''Weber number'' W. If R is of the order of magnitude of $(\sim)\epsilon^a$ and $W\sim\epsilon^b$, then $W\sim R^{b/a}$. The set of points $W=R^{b/a}$ is just a 1D curve in the 2D space (R,W), while the complete domain for which the EE is valid is likely to have the same dimensionality as the parameter space (R,W) itself, that is it should be a 2D domain.

Furthermore, each time the powers assigned to the parameters are altered it is, in principle, necessary to again go through the entire procedure or the SP derivation, and one can arrive at a different EE as a result. For example, Topper and Kawahara [16] considered two cases of an inclined-film flow (such a flow is specified by *three* global parameters; in addition to R and W, the inclination angle θ can be independently varied). In one case, they required the angle ϕ ($\equiv \pi/2-\theta$) of the film plane with the vertical to be small, $\phi\sim\epsilon$, and stipulated $R\sim\epsilon^1$ and $W\sim\epsilon^{-1}$ (in our definitions of R and W; see Sec. II below). As a result, they obtained an EE containing *both* dissipative and dispersive terms (and with all coefficients in the EE being ~ 1). However, their second case, for which they chose $\cot\theta\sim 1$, $R\sim 1$, and $W\sim\epsilon^{-1}$, resulted in an equation with *no* dispersive terms. (In the SP framework, it is only formally that the latter equation can be obtained from the former by omitting its dispersive term, and the only way to really ''justify'' the nondispersive equation is to repeat the entire derivation procedure starting all the way back from the NS equations.)

If a given inclined-film system is not close to any of these two parameter-space curves, the theory [16] is invalid. Logically, there are three possibilities: (i) the flow evolution cannot be (approximately) reduced to a single EE for all time, (ii) such a reduction *is* possible but the resulting EE is different from each of those obtained in Ref. [16], or (iii) the EE *coincides* with one of their two EEs, despite the different parameter curve. We are naturally led to the following questions: (i) Under what (parametric) conditions is an approximate description of the film flow possible (for all time) which can be reduced to a single EE? (ii) How can the set of all such EEs be characterized? These are the questions we pose and attempt to answer, for inclined-film flow, in this paper.

We note that one must distinguish between the *all-time* validity of an EE and the *limited* in time validity. This issue arises because, in the SP approach, the fixed powers of the small parameter are stipulated not only for the global parameters of the system, which do not depend on time, but also for the characteristic time and length scales. However, these characteristic scales can change with time as the *dissipative* system proceeds to the attractor; so, they are *instantaneous* parameters. Thus, the assumption of fixed scales is incorrect after a limited time has elapsed and, therefore, the SP derivation is invalid.

The rest of the paper is organized as follows. In Sec. II, we formulate the full NS problem. In Sec. III, we introduce an iterative perturbation procedure. It starts with the well-known (although typically unstable), waveless, ''Nusselt'' solution of the NS problem. The only principle necessary in deciding the iteration steps is the requirement that in the end, a *single* (and valid for all time) EE should be arrived at, with minimal simplification of exact equations. No dependencies are imposed on the internal parameters: the validity conditions (VCs) we obtain require that several quantities, which are certain products of powers of the internal parameters, be small—*independently* of one another. [So, *several* independent (small) parameters emerge in the derivation. Thus, the iterative procedure we introduce here is a variation of the ''multiparametric'' perturbation approach developed in Refs. [17–20].]

In Sec. IV, we arrive at the most general evolution equation (GEE) valid (provided certain restrictions on parameters are satisfied) for all time; any all-time-valid EE derivable with a conventional single-parameter approach necessarily coincides with one of just a few prototype equations which are certain truncations of the GEE. The GEE also has the least restrictive domain of validity; it is easy to obtain the domain of validity for each simplified EE, and it is a subdomain of the all-embracing (corresponding to the GEE) validity domain. Outside of the latter parametric domain, there is *no* single EE which could approximate the film evolution for all time. That this is the case is argued in Sec. V by analyzing the structure of possible correction terms of the EE through all orders of the iteration procedure. [In particular, as was shown in Refs. [21,22] —where a different derivation of the same EE was sketched—the *amplitudes* of inclined-film waves, which can be described (for all time) by a single EE, are necessarily *small*.] The paper is summarized in the last section. Some of the more technical considerations are relegated to the Appendix.

Simulation results for different versions of the inclined-film flow EEs we obtain here will be given elsewhere. [For some of those results (in particular, those showing good agreement with experiments [9]), see Ref. [4].]

II. EXACT NAVIER-STOKES PROBLEM

We consider a layer of an incompressible Newtonian liquid flowing down an inclined plane under the action of gravity. Our (Cartesian) coordinates are as follows: the $\overline{x}$ axis is normal to the plane and directed into the film, the $\overline{y}$ axis is in the spanwise direction, and the $\overline{z}$ axis is directed streamwise (the overbar here and below indicates a *dimensional* quantity). The corresponding components of velocity are $\overline{u}$, $\overline{v}$, and $\overline{w}$. We denote by $\overline{p}$ the pressure field in the film; the pressure of the ambient passive gas is neglected for simplicity.

The system is determined by the following independent (dimensional) parameters: the average thickness of the film $\overline{h}_0$; the liquid density $\overline{\rho}$, viscosity $\overline{\mu}$, and surface tension $\overline{\sigma}$; gravity acceleration $\overline{g}$; and the angle of the plane with the horizontal θ.

There is a well-known time-independent Nusselt's solution of the NS problem for the inclined film. The thickness of the Nusselt film is constant (hence, Nusselt's flow is also referred to as a ''flat-film'' solution). The only nonzero component of velocity is the streamwise one. It only changes across the film, starting from the zero value at the solid plane. The free-surface value $\overline{U}$ of the Nusselt velocity is $\overline{U}=\overline{g}\overline{h}_0^2\sin\theta/(2\overline{\nu})$ (where $\overline{\nu}=\overline{\mu}/\overline{\rho}$ is the kinematic viscosity). We nondimensionalize all quantities with units of measurement based on $\overline{\rho}$, $\overline{h}_0$, and $\overline{U}$ (we have, e.g., the following units: $\overline{h}_0$ for all coordinates, $\overline{U}$ for velocities, $\overline{h}_0/\overline{U}$ for time t, $\overline{\rho}\overline{U}^2$ for pressure, $\overline{\rho}\overline{U}^2\overline{h}_0$ for surface tension, etc.). We will see that exactly three independent *basic parameters* (BPs) appear in the *dimensionless* equations and boundary conditions of the problem; one can choose, e.g., the inclination angle θ, the Reynolds number $R\equiv\overline{h}_0\overline{U}/\overline{\nu}$ $[=\overline{g}\overline{h}_0^3\sin\theta/(2\overline{\nu}^2)]$, and the Weber number $W\equiv\sigma R/2$ $[=\overline{\sigma}/(\overline{\rho}\overline{g}\overline{h}_0^2\sin\theta)]$, as such BPs.

We can write the Navier-Stokes momentum equations in coordinates moving relative to the solid plane with a constant velocity V in the z direction (i.e., introducing $\tilde{z}=z-Vt$ and omitting the tilde) in the following dimensionless form (see, e.g., Ref. [23]):

$$u_t+uu_x+vu_y+wu_z-Vu_z$$
$$=-p_x-\frac{2}{R}\cot\theta+\frac{1}{R}(u_{xx}+u_{yy}+u_{zz}), \quad (1)$$

$$v_t+uv_x+vv_y+wv_z-Vv_z$$
$$=-p_y+\frac{1}{R}(v_{xx}+v_{yy}+v_{zz}), \quad (2)$$

$$w_t+uw_x+vw_y+ww_z-Vw_z$$
$$=-p_z+\frac{2}{R}+\frac{1}{R}(w_{xx}+w_{yy}+w_{zz}). \quad (3)$$

(The subscripts x, y, z, and t here and below denote the corresponding partial derivatives. We will see below that it is appropriate to choose $V=2$, the common phase velocity of all infinitesimally weak waves.) The continuity equation is

$$u_x+v_y+w_z=0. \quad (4)$$

The BCs are as follows. The no-slip conditions at the solid plane are

$$u=v=w=0 \quad (x=0). \quad (5)$$

The tangential-stress balance conditions at the free surface $x=h(y,z,t)$, the local film thickness, are

$$(v_x+u_y)(1-h_y^2)+2(u_x-v_y)h_y-(v_z+w_y)h_z$$
$$-(u_z+w_x)h_yh_z=0 \quad (x=h) \quad (6)$$

and

$$(u_z+w_x)(1-h_z^2)+2(u_x-w_z)h_z-(v_z+w_y)h_y$$
$$-(u_y+v_x)h_yh_z=0 \quad (x=h). \quad (7)$$

The normal-stress balance condition is

$$-p(1+h_y^2+h_z^2)^{3/2}+\frac{2}{R}[u_x+v_yh_y^2+w_zh_z^2$$
$$-(u_y+v_x)h_y+(v_z+w_y)h_yh_z$$
$$-(u_z+w_x)h_z](1+h_y^2+h_z^2)^{1/2}$$
$$=\sigma[h_{yy}(1+h_z^2)+h_{zz}(1+h_y^2)-2h_yh_zh_{zy}] \quad (x=h). \quad (8)$$

Finally, the kinematic condition at the free surface is

$$h_t+vh_y+wh_z-Vh_z=u \quad (x=h). \quad (9)$$

III. ITERATIVE PERTURBATION PROCEDURE

A. Minimal requirement of derivability

As was motivated in the Introduction, we are interested in the question of (approximate) reducibility of the above complicated description of inclined-film dynamics to a single EE. It appears that an *iterative* perturbation approach is appropriate for the general analysis of the problem.

Looking at the previously known, conventional (single-parameter expansion) derivations of EEs (valid each for its own particular curve in the parameter space; see the Introduction), it is clear that each of them in effect discards some terms of the NS momentum equations so that each of those essentially becomes an ordinary differential equation (ODE) in x, linear and with constant coefficients, that is easy to solve (with similar simplifications in BCs). When these solutions (for velocities in terms of film thickness) are substituted into the kinematic condition (9), the EE for thickness h (or, equivalently, for the thickness deviation $\eta\equiv h-1$) ensues. However, after all the quantities have been expanded in power of ϵ (see the Introduction), one has no control over which NS terms to omit; this is simply dictated by the expansion scheme of the SP approach. Thus, sometimes ''harmless'' terms are discarded; even if they were retained, one still would be able to solve for velocities, arriving, as a result, at a clearly *more general* EE.

Accordingly, our main idea in this paper is to look for a derivation in which the single postulated requirement would be that a maximally general EE for film thickness be arrived at in the end, so only those terms of the exact NS equations will be discarded which are clearly in the way of obtaining linear ODEs for velocities and pressure [we call this principle the ''minimal requirement of derivability'' (MRD)]. In this way we obtain approximate solutions for the deviations of exact solutions from the ''seed'' Nusselt's fields [e.g., the approximation w_0 to $\tilde{w}_0\equiv w-w_N$, where the Nusselt w_N is known; see Eq. (12) below]. For some parameter values, this immediately leads to an EE whose approximation of exact evolution is good for all time (here, as was mentioned in the Introduction, we are only interested in such all-time-valid EEs; see, e.g., Ref. [4] for a further discussion of their difference from the EEs whose validity is *limited* in time). In other cases, however, the procedure must be repeated, with the refined solutions, such as w_N+w_0, playing the seed role that was played by the Nusselt solutions on the original stage (so that at the second iteration stage one determines the approximation w_1 to $\tilde{w}_1\equiv w-w_N-w_0$, etc.). Thus, ours is an *iterative* perturbation approach, which is known even in general to be an alternative to the *expansion* method (see, e.g., Ref. [24]).

Estimating all members of NS problem equations in terms of parameters, the requirement that the discarded members be much smaller than those retained leads to the parametric conditions for our derivation to be valid (see Appendix A). Thus, the method yields (i) the evolution equation for film thickness, (ii) explicit expressions for velocity components and pressure in terms of the film thickness, and (iii) the parametric validity conditions of the theory.

It is known (see Ref. [4] and references therein) that no single EE description can exist globally in time for those parametric regimes of inclined-film flow which lead to the eventual amplitude of surface waves being ''large''—comparable to the average film thickness. But in the present communication, as was mentioned above, we are interested exactly in the *large-time* behavior, when the system is already close to the attractor, and we want a *single* EE description of the wavy film dynamics. Therefore, in addition to the

MRD principle, we can use from the very beginning—in order to simplify our derivation—the requirement that the amplitude A of the film thickness deviation, $A(t)\equiv\max|\eta|$, be small (for all time); with

$$h=1+\eta, \tag{10}$$

we have

$$\max|\eta(y,z,t)|=A\ll 1. \tag{11}$$

[Note that A can depend upon time; in such cases, we say that the parameter is a *local* parameter, in contrast to the time-independent ''global'' basic parameters θ, R, and W.] However, unlike the conventional derivations, we do not have to postulate that the characteristic length scales in the film plane are large (the long-wave assumption); rather, this will *follow* from the MRD principle.

B. Nusselt's solution

The above-mentioned Nusselt solution (of the NS problem), which is steady and uniform along the film (i.e., in the streamwise and spanwise directions), is as follows. The dimensionless Nusselt streamwise velocity w_N is

$$w_N(x)=2x-x^2. \tag{12}$$

This clearly satisfies the z NS equation

$$w_{Nxx}=-2, \tag{13}$$

with the boundary conditions

$$w_{Nx}=0 \quad (x=1) \tag{14}$$

and

$$w_N=0 \quad (x=0). \tag{15}$$

Similarly, the Nusselt pressure

$$p_N=\frac{2}{R}(\cot\theta)(1-x) \tag{16}$$

is the solution of the x NS equation

$$p_{Nx}=-\frac{2}{R}\cot\theta, \tag{17}$$

with the boundary condition

$$p_N=0 \quad (x=1). \tag{18}$$

Finally, the Nusselt normal and spanwise velocities are

$$u_N=0 \quad \text{and} \quad v_N=0. \tag{19}$$

At sufficiently large Reynolds number, the destabilizing effect of inertia overcomes the stabilizing influence of gravity so that the Nusselt solution loses its stability. (An analysis later in this section leads to $R>(5/4)\cot\theta$, the well-known criterion for instability.) As a result, the film is not uniform any more and also changes with time.

C. First iteration step

We represent the exact velocities and pressure in the form of sums of Nusselt's solutions and the deviations from those:

$$w=w_N+\tilde{w}_0, \quad u=u_N+\tilde{u}_0,$$
$$v=v_N+\tilde{v}_0, \quad p=p_N+\tilde{p}_0, \tag{20}$$

where tildes indicate the deviations from the Nusselt solutions. First, we consider the z-momentum NS equation. We substitute the expressions (20) for velocities and pressure into Eq. (3). The resulting (exact) equation can be written in the form

$$\tilde{w}_{0xx}=R\tilde{p}_{0z}-\nabla^2\tilde{w}_0+R\tilde{w}_{0t}+R\tilde{u}_0(w_N+\tilde{w}_0)_x+R\tilde{v}_0\tilde{w}_{0y}$$
$$+R(w_N+\tilde{w}_0-V)\tilde{w}_{0z}, \tag{21}$$

where $\nabla^2=\partial^2/\partial y^2+\partial^2/\partial z^2$. In accordance with the MRD principle, as was discussed above, in order to obtain a solvable ODE in x, we have to discard all the terms on the right-hand side (RHS) of Eq. (21), since every one of those contains unknown quantities. This yields a simplified equation for the velocity $\tilde{w}_0$, whose solution we denote by w_0:

$$w_{0xx}=0. \tag{22}$$

We call $\tilde{w}_1$ the error in the approximation of $\tilde{w}_0$ by w_0:

$$\tilde{w}_0=w_0+\tilde{w}_1. \tag{23}$$

It is clear [see Eqs. (10) and (11)] that the characteristic x length scale X is of O_M of 1 $(X\sim 1)$, i.e. (since $\partial/\partial x \sim 1/X$),

$$\frac{\partial}{\partial x}\sim 1. \tag{24}$$

We denote by Y and Z the characteristic length scales in the spanwise and streamwise directions respectively, so that

$$\frac{\partial}{\partial y}\sim\frac{1}{Y} \tag{25}$$

and

$$\frac{\partial}{\partial z}\sim\frac{1}{Z}. \tag{26}$$

Similarly, denoting by T the characteristic time scale, we have

$$\frac{\partial}{\partial t}\sim\frac{1}{T}. \tag{27}$$

Also, clearly, $\partial^2/\partial x^2\sim 1$, $\partial^2/\partial z^2\sim 1/Z^2$, $\partial/\partial y^2\sim 1/Y^2$, etc.

The characteristic magnitude of each of the neglected terms on the RHS of Eq. (21)—estimated by replacing $\tilde{w}_0$ with w_0, etc.—should be smaller than that of the term on the left-hand side, w_{0xx}: $w_0/Z^2\ll w_0/X^2$ and $w_0/Y^2\ll w_0/X^2$. Hence, we have the following restrictions on the (instantaneous) length scales for the validity of our theory:

$$\frac{1}{Y^2}\ll 1 \tag{28}$$

and

$$\frac{1}{Z^2}\ll 1. \tag{29}$$

Thus, as a consequence of our derivability requirement, we have obtained the longwave conditions, which are rather *postulated* in the conventional ''lubrication'' or ''long-wave'' derivations.

Similarly, using Eqs. (26) and (27), the conditions that the advective inertial term containing w_0, that is the term RVw_{0z}, and the time-derivative term Rw_{0t}, be both smaller than w_{0xx}, lead to the following requirements on the parameters:

$$\frac{R}{Z}\ll 1 \tag{30}$$

and

$$\frac{R}{T}\ll 1. \tag{31}$$

Henceforth, we confine our consideration to the film configurations which satisfy the conditions (28)–(31). Note that since the local parameters *A, Z, Y,* and *T* can change with time, they may cease to satisfy the validity conditions [such as Eq. (11) and Eqs. (28)–(31)] at a certain stage of the evolution. In such cases, clearly, the single EE description would be valid only for a limited time that these (and some other, additional conditions appearing in Appendix A) still hold. Later, we will determine domains in the space of global parameters for which such a violation of VCs never happens—so that the single EE approximation is valid globally rather than merely locally in time.

The boundary condition on $\tilde{w}_0$, at $x=h$, given by the tangential-stress balance equation (7), is written in the form

$$\tilde{w}_{0x}=-w_{Nx}-\tilde{u}_{0z}+[2(\tilde{w}_{0z}-\tilde{u}_{0x})\eta_z+(\tilde{v}_{0z}+\tilde{w}_{0y})\eta_y + (\tilde{u}_{0y}+\tilde{v}_{0x})\eta_y\eta_z](1-\eta_y^2)^{-1} \quad (x=h). \tag{32}$$

According to the MRD principle, we have to drop all the terms containing the unknown quantities [those denoted by letters with tilde] on the RHS. Also, using the smallness of the surface deviation (11), we will everywhere transfer the boundary conditions from the true boundary $x=h$ to a convenient ''boundary'' $x=1$ by expanding all quantities in Taylor series around $x=1$ (cf. Ref. [24]), such as

$$\tilde{w}_{0x}(x=h). \tag{33}$$

Noting that $w_{Nx}(x=1)=0$, we have the simplified boundary condition (for the *approximation* w_0 of the exact quantity $\tilde{w}_0$):

$$w_{0x}=-w_{Nxx}\eta=2\eta \quad (x=1). \tag{34}$$

Finally, the no-slip condition requires

$$w_0=0 \quad (x=0). \tag{35}$$

The solution of Eq. (22) with the boundary conditions (34) and (35) is clearly

$$w_0=2\eta x. \tag{36}$$

Note that from Eqs. (12) and (36),

$$w_{Nx}+w_{0x}=0 \quad (x=h), \tag{37}$$

which we will use below. We also observe that

$$w_0\sim A(\ll w_N\sim 1). \tag{38}$$

Next, consider the incompressibility equation [see Eq. (4)] in the form

$$\tilde{u}_{0x}=-\tilde{v}_{0y}-w_{0z}-\tilde{w}_{1z}. \tag{39}$$

(Note that w_N does not make any contribution to this equation, since w_N does not depend on *z*.) Dropping the terms with unknowns $\tilde{v}_0$ and $\tilde{w}_1$ on the RHS, we obtain an equation for the approximation u_0:

$$u_{0x}=-w_{0z}=-2x\eta_z. \tag{40}$$

The no-slip BC (5) requires

$$u_0=0 \quad (x=0), \tag{41}$$

and one readily obtains the solution

$$u_0=-x^2\eta_z. \tag{42}$$

We note that

$$u_0\sim\frac{A}{Z}\ll 1. \tag{43}$$

We denote the deviation of the exact solution $\tilde{u}_0$ from the approximation u_0 by $\tilde{u}_1$, so that

$$\tilde{u}_0=u_0+\tilde{u}_1. \tag{44}$$

The (*x*-momentum) NS equation for the deviation of pressure from the Nusselt solution p_N is

$$\tilde{p}_{0x}=\frac{1}{R}(u_{0xx}+u_{0yy}+u_{0zz})+\frac{1}{R}(\tilde{u}_{1xx}+\tilde{u}_{1yy}+\tilde{u}_{1zz}) -(u_0+\tilde{u}_1)_t-(u_0+\tilde{u}_1)(u_0+\tilde{u}_1)_x-\tilde{v}_0(u_0+\tilde{u}_1)_y -(w_N+w_0+\tilde{w}_1-V)(u_0+\tilde{u}_1)_z \tag{45}$$

[where we have used Eq. (17)]. In this equation, we have to drop the terms containing the unknowns $\tilde{v}_0$, $\tilde{w}_1$, or $\tilde{u}_1$. Next, it is easy to show that the u_{0xx} term [which is $\sim u_0$ due to Eq. (24)] on the RHS is much larger than the other terms. Namely, $u_{0yy}\sim u_0/Y^2\ll u_0\sim u_{0xx}$ due to Eq. (28) and $u_{0zz}\sim u_0/Z^2\ll u_0\sim u_{0xx}$ due to Eq. (29). Similarly, $Ru_{0t}\sim(R/T)u_0\ll u_0\sim u_{0xx}$ by making use of Eq. (31), and $R(w_N+w_0-V)u_{0z}\sim Ru_{0z}\sim(R/Z)u_0\ll u_0\sim u_{0xx}$ due to Eq.

(30). Finally, $u_0 u_{0x} \ll u_{0x} \sim u_{0xx}$, since $u_0 \ll 1$ [see Eq. (43)]. Thus the simplified equation (for the *approximation* p_0) is

$$p_{0x} = \frac{1}{R} u_{0xx} = -\frac{2}{R} \eta_z . \tag{46}$$

The BC at $x=h$ [see the normal-stress balance condition, Eq. (8)] is

$$\begin{aligned} \tilde{p}_0 = & -p_N + \frac{2}{R}\{(u_0+\tilde{u}_1)_x + \tilde{v}_{0y}\eta_y^2 + (w_0+\tilde{w}_1)_z \eta_z^2 \\ & -[(u_0+\tilde{u}_1)_y + \tilde{v}_{0x}]\eta_y + [\tilde{v}_{0z} + (w_0+\tilde{w}_1)_y]\eta_y \eta_z \\ & -[(u_0+\tilde{u}_1)_z + \tilde{w}_{1x}]\eta_z\}(1+\eta_y^2+\eta_z^2)^{-1} \\ & -\sigma[\eta_{yy}(1+\eta_z^2) + \eta_{zz}(1+\eta_y^2) - 2\eta_y\eta_z\eta_{yz}] \\ & \times(1+\eta_y^2+\eta_z^2)^{-3/2} \quad (x=h) \end{aligned} \tag{47}$$

[where we have used Eq. (37) to eliminate w_{Nx} and w_{0x}]. We have to drop those terms containing $\tilde{v}_0$, $\tilde{w}_1$, or $\tilde{u}_1$. In addition, a number of known terms are estimated to be smaller than the term with u_{0x}. Namely, by using the estimates of w_0 and u_0 [Eqs. (38) and (43)], $w_{0y}\eta_y\eta_z \sim (A/Z)(A/Y)^2 \ll A/Z \sim u_0 \sim u_{0x}$. Similarly, $w_{0z}\eta_z^2 \sim (A/Z) \times (A^2/Z^2) \ll u_{0x}$. Also, $u_{0y}\eta_y \sim u_0 A/Y^2 \ll u_{0x}$ and $u_{0z}\eta_z \sim u_0 A/Z^2 \ll u_{0x}$. Finally, one can see that each of the other known nonlinear terms can be neglected since $\eta_z^2 \sim A^2/Z^2 \ll 1$, $\eta_y^2 \sim A^2/Y^2 \ll 1$, and $\eta_y\eta_z\eta_{yz} \sim A^3/(Y^2Z^2) \ll 1$. We also note that the entire Taylor expansion for $p_N(x=1+\eta)$ consists of just one term,

$$p_N(h) = p_{Nx}(1)\eta. \tag{48}$$

With the known expressions for p_N, w_0, and u_0 (see above), and using Taylor series about $x=1$, truncated to the first nonzero term, for all the quantities, one obtains the BC

$$\begin{aligned} p_0 &= -p_{Nx}\eta + \frac{2}{R}u_{0x} - \sigma\nabla^2\eta \\ &= \frac{2}{R}(\cot\theta)\eta - \frac{4}{R}\eta_z - \sigma\nabla^2\eta \quad (x=1). \end{aligned} \tag{49}$$

The solution of Eq. (46) with the boundary condition (49) is

$$Rp_0 = 2(\cot\theta)\eta - 2\eta_z(1+x) - 2W\nabla^2\eta. \tag{50}$$

We observe that

$$Rp_0 \sim \max\left(\cot\theta, \frac{1}{Z}, \frac{W}{Z^2}, \frac{W}{Y^2}\right)A. \tag{51}$$

As usual, we call $\tilde{p}_1$ the difference between $\tilde{p}_0$ and its approximation p_0:

$$\tilde{p}_0 = p_0 + \tilde{p}_1 . \tag{52}$$

Finally, consider the y NS equation,

$$\begin{aligned} \tilde{v}_{0xx} = & R(p_0+\tilde{p}_1)_y - \nabla^2\tilde{v}_0 + R\tilde{v}_{0t} + R(u_0+\tilde{u}_1)\tilde{v}_{0x} \\ & + R\tilde{v}_0\tilde{v}_{0y} + R(w_N + w_0 + \tilde{w}_1 - V)\tilde{v}_{0z} . \end{aligned} \tag{53}$$

Dropping the unknown terms on the RHS, we obtain the simplified equation:

$$v_{0xx} = Rp_{0y} = 2(\cot\theta)\eta_y - 2(1+x)\eta_{yz} - 2W\nabla^2\eta_y , \tag{54}$$

where we have used the expression (50) for p_0 in terms of η to get the RHS in the explicit form.

The BC on $\tilde{v}_0$, at $x=h$, is [see Eq. (6)]

$$\begin{aligned} \tilde{v}_{0x} = & -(u_0+\tilde{u}_1)_y + \{2[\tilde{v}_{0y} - (u_0+\tilde{u}_1)_x]\eta_y \\ & + [(u_0+\tilde{u}_1)_z + \tilde{w}_{1x}]\eta_y\eta_z \\ & + [\tilde{v}_{0z} + (w_0+\tilde{w}_1)_y]\eta_z\}(1-\eta_y^2)^{-1} \quad (x=h), \end{aligned} \tag{55}$$

where we have again used Eq. (37) to eliminate $w_{Nx} + w_{0x}$. Omitting the terms with tildes and performing the, by now, familiar estimates of the known terms on the RHS, we see that the term u_{0y} is larger than any other term, so that the simplified BC at $x=1$ is

$$v_{0x} = -u_{0y} = \eta_{zy} \quad (x=1), \tag{56}$$

where the expression (42) for u_0 has been used. Finally, the no-slip condition

$$v_0 = 0 \quad (x=0) \tag{57}$$

is to be satisfied. The solution of Eq. (54) with the BCs (56) and (57)—as can be readily checked by direct substitution—is

$$v_0 = (\cot\theta\,\eta_y - W\nabla^2\eta_y)(x^2-2x) - \eta_{zy}\left(\frac{x^3}{3} + x^2 - 4x\right). \tag{58}$$

This yields the following estimate for v_0:

$$v_0 \sim \max\left[\frac{\cot\theta}{Y}, \frac{W}{Y^3}, \frac{W}{YZ^2}, \frac{1}{YZ}\right]A. \tag{59}$$

As usual,

$$\tilde{v}_0 = v_0 + \tilde{v}_1 . \tag{60}$$

Substituting Eq. (60) into Eq. (39) we observe that, in obtaining the simplified equation (40) for u_{0x}, we have in fact discarded v_{0y}. This requires

$$v_{0y} \sim \frac{v_0}{Y} \ll u_{0x} \sim u_0 \tag{61}$$

(which will be used below). By using here the estimates (43) for u_0 and (59) for v_0, we arrive at the VCs $\max[Z\cot\theta/Y^2, WZ/Y^4, W/(Y^2Z)] \ll 1$. These are a subset of the complete set of (instantaneous) VCs obtained in Appendix A [see Eq. (A10)]. It is straightforward to verify that each discarded term in every step of our procedure is small as a consequence of those VCs.

The kinematic condition [Eq. (9)] at $x=h$ becomes

$$\eta_t+(v_0+\tilde{v}_1)\eta_y+(w_N+w_0+\tilde{w}_1-V)\eta_z=u_0+\tilde{u}_1. \tag{62}$$

Dropping the unknown terms with tildes and the smaller terms $v_0\eta_y\sim(v_0/Y)\eta\ll u_0$ [see Eq. (61)] and $w_0\eta_z\sim A^2/Z\ll u_0$, we have

$$\eta_{t_0}+(w_N-V)\eta_z=u_0, \tag{63}$$

where we have introduced the ''fast'' time t_0, such that $\partial_t=\partial_{t_0}+\partial_{t_1}\approx\partial_{t_0}$. Using the Taylor expansions for w_N and u_0, we obtain, at $x=1$,

$$\eta_{t_0}+(2-V)\eta_z=0. \tag{64}$$

Choosing $V=2$, we can eliminate the fast-time undulations (which are clearly due to the uniform translation of the wave, with no change in its shape):

$$\eta_{t_0}=0. \tag{65}$$

Thus, the leading approximation determines the velocity of a reference frame in which film thickness does not change on the fast time scale. However, it will change with the slower time t_1. In order to obtain this slower-time evolution of the film thickness, one needs to consider the next approximation for the velocities and pressure. From now on, we fix

$$V=2. \tag{66}$$

Then, in view of Eq. (65), $\partial_t=\partial_{t_1}$.

D. Second iteration

We now proceed to consider the ''corrections'' $\tilde{w}_1$, $\tilde{u}_1$, $\tilde{p}_1$, and $\tilde{v}_1$ for the velocities and pressure. By substituting

$$\tilde{w}_0=w_0+\tilde{w}_1,\quad \tilde{u}_0=u_0+\tilde{u}_1,$$
$$\tilde{v}_0=v_0+\tilde{v}_1,\quad \tilde{p}_0=p_0+\tilde{p}_1 \tag{67}$$

into the z NS equation (21), and taking into account $w_{0xx}=0$ (22), we have the exact equation

$$\begin{aligned}\tilde{w}_{1xx}=&-w_{0xx}+Rp_{0z}-\nabla^2w_0+Rw_{0t}+Ru_0(w_N+w_0)_x\\&+Rv_0w_{0y}+R(w_N+w_0-2)w_{0z}\\&+[\text{terms containing }\tilde{w}_1,\ \tilde{u}_1,\tilde{v}_1,\text{ or }\tilde{p}_1].\end{aligned} \tag{68}$$

Performing our standard simplification procedure, i.e., discarding the unknown terms (containing tildes) and small terms, and also taking into account that $w_0\ll w_N$ [see Eq. (38)] and $v_{0y}\ll u_0$ [see Eq. (61)], the simplified equation for the approximation w_1 is

$$\begin{aligned}w_{1xx}&=Rp_{0z}-\nabla^2w_0+Ru_0w_{Nx}+Rw_Nw_{0z}-2Rw_{0z}+Rw_{0t}\\&=2(\cot\theta\,\eta_z-W\nabla^2\eta_z)-2(x+1)\eta_{zz}\\&\quad-2x\nabla^2\eta+(2x^2-4x)R\eta_z+2xR\eta_t,\end{aligned} \tag{69}$$

where we have used the known expressions (in terms of surface deviation η; see the preceding section) for the first-iteration approximations p_0, w_0, and u_0. [For analogous equations of the general, nth, iteration step, see Appendix C of Ref. [25] (the present paper extended by two appendices: Appendix B and Appendix C).]

The BC on $\tilde{w}_1$, at $x=h$, comes from Eq. (7):

$$\begin{aligned}\tilde{w}_{1x}=&-u_{0z}+2w_{0z}\eta_z-2u_{0x}\eta_z+w_{0y}\eta_y+v_{0z}\eta_y\\&+(u_{0y}+v_{0x})\eta_y\eta_z\\&+[\text{terms containing }\tilde{w}_1,\ \tilde{u}_1,\tilde{v}_1,\text{ or }\tilde{p}_1]\quad(x=h),\end{aligned} \tag{70}$$

where we have taken into account Eq. (37). Continuing to use the simplification procedure established in the previous section, we arrive at the boundary condition

$$w_{1x}=-u_{0z}=\eta_{zz}\quad(x=1). \tag{71}$$

All other terms on the RHS of Eq. (70) are readily estimated in our usual way to be smaller than u_{0z} [Eq. (61) is useful in estimating terms containing v]. The no-slip condition is

$$w_1=0\quad(x=0). \tag{72}$$

The solution of the problem (69), (71), and (72) is

$$\begin{aligned}w_1=&(\cot\theta\,\eta_z-W\nabla^2\eta_z)(x^2-2x)\\&+\left(\frac{x^4}{6}-\frac{2}{3}x^3+\frac{4}{3}x\right)R\eta_z+\left(5x-\frac{2}{3}x^3-x^2\right)\eta_{zz}\\&+\left(x-\frac{x^3}{3}\right)\eta_{yy}+\left(\frac{x^3}{3}-x\right)R\eta_t,\end{aligned} \tag{73}$$

which can be verified by direct substitution into the problem equations. Note that all the terms of w_1 are estimated to be *quadratic* in the local parameters (A10); one can show as a generalization (see Appendix C of Ref. [25]) that w_n is of the power $(n+1)$, and similarly for u_n, v_n, and p_n.

Taking into account $u_{0x}=-w_{0z}$ [see Eq. (40)], the incompressibility condition yields the equation for $\tilde{u}_1$,

$$\tilde{u}_{1x}=-w_{1z}-\tilde{w}_{2z}-v_{0y}-\tilde{v}_{1y}, \tag{74}$$

where we have expressed $\tilde{w}_1$ as $\tilde{w}_1=w_1+\tilde{w}_2$. Dropping the unknown terms, we obtain an equation for the approximation u_1:

$$\begin{aligned}u_{1x}&=-w_{1z}-v_{0y}\\&=-(x^2-2x)(\cot\theta\nabla^2\eta-W\nabla^4\eta)\\&\quad-\left(\frac{x^4}{6}-\frac{2}{3}x^3+\frac{4}{3}x\right)R\eta_{zz}\\&\quad-\left(5x-\frac{2}{3}x^3-x^2\right)\nabla^2\eta_z-\left(\frac{x^3}{3}-x\right)R\eta_{tz}\end{aligned} \tag{75}$$

with the BC

$$u_1=0 \quad (x=0). \tag{76}$$

It is easy to verify that

$$u_1=\left(\frac{x^3}{3}-x^2\right)[W\nabla^4\eta-\cot\theta\nabla^2\eta]-\left(\frac{x^5}{30}-\frac{x^4}{6}+\frac{2}{3}x^2\right)R\,\eta_{zz}+\left(\frac{x^4}{6}+\frac{x^3}{3}-\frac{5}{2}x^2\right)\nabla^2\eta_z-\left(\frac{x^4}{12}-\frac{x^2}{2}\right)R\,\eta_{tz} \tag{77}$$

is the solution. We note that the complete set of VCs (A10) obtained in Appendix A guarantees that all the terms discarded in obtaining solvable ODEs for w_1 and u_1 are small in comparison with (the biggest of) those terms that are retained.

At this point, we could proceed to solve the x NS and y NS equations for the pressure and velocity corrections $\tilde{p}_1$ and $\tilde{v}_1$, respectively. However, these corrections are not needed for obtaining the second-iteration EE. (One only needs the pressure and velocity corrections for the *later* iteration stages. These corrections are calculated in Appendix B of Ref. [25].)

IV. THE DISPERSIVE-DISSIPATIVE EVOLUTION EQUATION

The (exact) kinematic condition (9) at $x=h$ can be written in the form

$$\eta_t+(v_0+\tilde{v}_1)\eta_y+(w_N+w_0+w_1+\tilde{w}_2-2)\eta_z=u_0+u_1+\tilde{u}_2 \quad (x=1+\eta), \tag{78}$$

where we have used $V=2$ (66). Dropping the terms containing unknown velocities (those with tildes), and using the Taylor series to relate the velocity components at $x=h$ to those at $x=1$, we have

$$\eta_t+(w_{Nx}\eta+w_0)\eta_z=u_{0x}\eta+u_1 \quad (x=1). \tag{79}$$

In Eq. (79), we have dropped the terms $v_0\eta_y$ and $w_1\eta_z$ as they are smaller than u_1 [see Eq. (75)]. Also, recall that $w_{Nx}(x=1)=0$ [see Eq. (12)]. Using the expressions (36), (42), and (77) for w_0, u_0, and u_1, we obtain

$$\left[\eta-\frac{5}{12}R\,\eta_z\right]_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{2}{3}\cot\theta\,\eta_{yy}+\frac{2}{3}W\nabla^4\eta+2\nabla^2\eta_z=0 \tag{80}$$

where, by definition,

$$\delta\equiv\frac{4}{5}R-\cot\theta. \tag{81}$$

However, the term $\propto R\,\eta_{tz}\sim(R/Z)\eta_t\ll\eta_t$, since $R/Z\ll 1$ [see Eq. (A10)]. Dropping this small term, we have

$$\eta_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{2}{3}\cot\theta\,\eta_{yy}+\frac{2}{3}W\nabla^4\eta+2\nabla^2\eta_z=0. \tag{82}$$

Simple linear-stability analysis can reveal the dynamical role of some terms here. Assuming an infinitesimally small disturbance in the form of a normal mode, $\eta\propto\exp(st-iwt)\exp i(jy+kz)$, it readily follows from the linearized version of Eq. (82) that

$$s=\frac{2}{3}[\delta k^2-(\cot\theta)j^2-W(k^4+2j^2k^2+j^4)] \tag{83}$$

and

$$\omega=-2k(k^2+j^2). \tag{84}$$

Here s is the growth (or decay) rate for the disturbance and so the third, fourth, and fifth terms in Eq. (82), which give rise to growth (or decay), are *dissipative* [considering the destabilizing term (the one with δ) as a *negative* dissipation]. In contrast, the last term in Eq. (82) only makes a contribution to the (real) frequency ω, rather than to the growth rate s, i.e., it does not lead to growth or decay of disturbances. Thus, this (third-derivative) term is *dispersive*.

Clearly, for instability to develop (i.e., for $s>0$), we need

$$\delta>0, \tag{85}$$

a condition we assume fulfilled from now on. This yields the so-called critical value R_c of the Reynolds number, $R_c=(4/5)\cot\theta$, at which the instability sets in.

One can see from the above derivation of the dispersive-dissipative EE (82) that the destabilizing (third) term originates from the inertia terms of the NS equations. The (stabilizing) fourth and fifth terms are due, respectively, to hydrostatic and capillary (i.e., surface-tension) parts of the pressure. Finally, the last, odd-derivative term is due to the viscous part of the pressure. Such a purely *dispersive* term also appeared in the EE obtained by Topper and Kawahara [16] for an almost vertical plane; they used the small angle of the plane with the vertical as their (single) perturbation parameter (see also the discussion in the Introduction of the present paper). Our derivation shows that assumption to be unnecessary. In particular, for the vertical film $\cot\theta=0$, and Eq. (82) becomes

$$\eta_t+4\eta\eta_z+\frac{8}{15}R\,\eta_{zz}+\frac{2}{3}W\nabla^4\eta+2\nabla^2\eta_z=0. \tag{86}$$

Although an equation of this structure (but with arbitrary coefficients) was postulated as a model equation in Ref. [26], it cannot be obtained from the derivation of Topper and Kawahara [16]; since their small parameter is proportional to $\cot\theta$, it becomes zero for the vertical case, and the Reynolds number (also proportional to the small parameter in that SP derivation) vanishes, which, clearly, cannot correspond to any flow at all.

The (infinite-dimensional) dynamical system governed by the *dissipative* equation (82) essentially ''forgets'' initial conditions as it evolves towards an attractor. There may be fluctuations on the attractor, but there is no systematic change in time. Clearly, then the amplitude-decreasing, stabilizing term must balance the destabilizing one (the latter tends to increase the deviation amplitude). So the two dissipative terms are necessarily of the same order of magnitude on the attractor.

As to the magnitude of the dispersive term relative to that of the dissipative terms, there can occur, depending on location in the parameter space, each of the following three possibilities: (i) these terms are of the same O_M, (ii) the dispersive term is small (and then the amplitude is determined by the balance of the nonlinear term with the dissipative terms), and (iii) the dissipative terms are small (and the nonlinear term balances the dispersive one). In the first case, continuing the iteration process would lead to small corrections to the terms which cannot significantly change the evolution. In the second case, the small dispersive term can be omitted with a negligible effect, so the corrections would be again immaterial.

But in the third case, when dissipation is small, the situation is very different. Discarding the dissipative terms leads to a 2D Korteweg–deVries (KdV) equation which was simulated numerically in Ref. [27]. The KdV equation is purely dispersive and never forgets the initial conditions. It has a one-parameter family of axisymmetric solutions which are traveling solitons, similar to the well-known 1D KdV case. Depending on the initial state, there may be solitons of different length scales (and therefore moving with different speeds) in the final state. However, if the small dissipative terms are present, they will slowly change the initial soliton of an arbitrary length scale. It will evolve along the soliton family until the length scale is attained which provides for the balance between the two dissipative terms (this effect was first studied for the 1D case in Ref. [28]; see also Refs. [29–31]). Therefore, the dissipative terms, even when *small*, are important; they determine the length scale of the solution.

However, only the *largest-magnitude* terms are guaranteed to be correct in the above beginning-iteration derivation; as for the smaller terms, further iterations might yield significant corrections to them. We consider this question (of higher iterations and corrections to small dissipative terms) in the next section. It turns out (perhaps, surprisingly) that such corrections can be important *locally* under some parametric conditions, but that no (single) corrected EE can approximate the evolution for all time. Equation (82) is thus the most general of those EEs that can be valid *globally* in time—under appropriate parametric restrictions, which can be completely determined only with the analysis of higher iterations of the NS problem, as is done in the next section. For the rest of this section, we continue the consideration of the GEE (82).

From the condition $\delta > 0$ [Eq. (85)], it follows that

$$R > \frac{5}{4}\cot\theta > \cot\theta. \tag{87}$$

From Eqs. (82) and (83), it is clear that, in order for instability to develop, we need $\delta\eta_{zz} > (\cot\theta)\eta_{yy}$. Using the y and z length scales, this yields, $Y^2/Z^2 > (\cot\theta)/\delta$. Noting that either $\delta \ll R \sim R_c \sim \cot\theta$ or $\delta \sim R$, we see that $Y \gg Z$ or $Y \sim Z$, except perhaps for $R \gg \cot\theta$. For simplicity, we assume that $Z \leqslant Y$, which seems to be the case in all experiments we know about. Then

$$L \equiv \min(Z,Y) = Z. \tag{88}$$

With this and the condition (87), the VCs (A15) reduce to

$$\max\left[A, \frac{1}{L^2}, \frac{R}{L}, \frac{W}{L^3}\right] \ll 1. \tag{89}$$

These are conditions of *instantaneous* validity; they involve the local (i.e., instantaneous) parameters $A(t)$ and $L(t)$.

As was discussed above, due to the dissipativeness of the EE (82), the system evolves towards an attractor, and in the asymptotic limit of large times we have $A(t) = \text{const} \equiv A_a$ and $L(t) = \text{const} \equiv L_a$. Since (similar to Refs. [32,17]) the destabilizing inertia term should be of the same O_M as the stabilizing, capillary one, i.e., $\delta\eta_{zz}(\sim\delta A/L_a^2) \sim W\nabla^4\eta$ $(\sim WA/L_a^4)$, the (dimensionless) characteristic length scale *at large times* L_a can be taken to be

$$L_a = \left(\frac{W}{\delta}\right)^{1/2}. \tag{90}$$

Similarly, the asymptotic magnitude of the characteristic amplitude A_a is determined by the balance between the nonlinear ''advective'' term and either the dispersive term or the capillary one (whichever is larger): $A_a = \max(W/L_a^3, 1/L_a^2)$. Using these asymptotic values of parameters, the conditions (89) can be written as $\max(W/L_a^3, R/L_a, 1/L_a^2) \ll 1$. Noting that, in view of Eq. (90), $W/L_a^3 = \delta/L_a$ and [see Eq. (81)] $R = (5/4)(\delta + \cot\theta) > \delta$, so that $W/L_a^3 < R/L_a$, we can simplify the VC to $\max(R/L_a, L_a^{-2}) \ll 1$; in terms of the basic parameters,

$$\alpha \equiv \frac{1}{L_a^2} = \frac{\delta}{W} \ll 1, \quad \beta \equiv \frac{R}{L_a} = R\left(\frac{\delta}{W}\right)^{1/2} \ll 1. \tag{91}$$

In the next section, it is shown that we also need

$$\gamma \equiv \frac{\max(R, R^3)}{W} \ll 1 \tag{92}$$

(otherwise, the dissipative terms contributed by higher iterations can become significant, and the evolution cannot be all-time describable by a single EE). All three parameters, α, β, and γ, are small if (recall that $\delta < R$)

$$\alpha_R \equiv \frac{R}{W} = \frac{\overline{\rho g^2}\,\overline{h}_0^5 \sin^2\theta}{2\overline{\sigma}\overline{\nu}^2} \ll 1 \tag{93}$$

and

$$\beta_R \equiv \frac{R^{3/2}}{W^{1/2}} = \left(\frac{\overline{\rho}\overline{h}_0^{11}}{8\overline{\sigma}}\right)^{1/2}\left(\frac{\overline{g}^2 \sin^2\theta}{\overline{\nu}^3}\right) \ll 1. \tag{94}$$

So, if we are in the domain of the space of basic parameters which satisfies the condition $\max(\alpha_R, \beta_R) \ll 1$ [or a bit more general, but less simple condition $\max(\alpha, \beta, \gamma) \ll 1$], then Eq. (82) is good *for all time* [provided that the initial amplitude and length scale satisfy the conditions (89)]. Therefore, we call such conditions the ''global'' VCs.

We can transform the GEE (82) to a ''canonical'' form—which contains only two ''tunable'' constants—by rescaling the variables with appropriate units:

$$\eta = N\tilde{\eta}, \quad z = L_a\tilde{z},$$

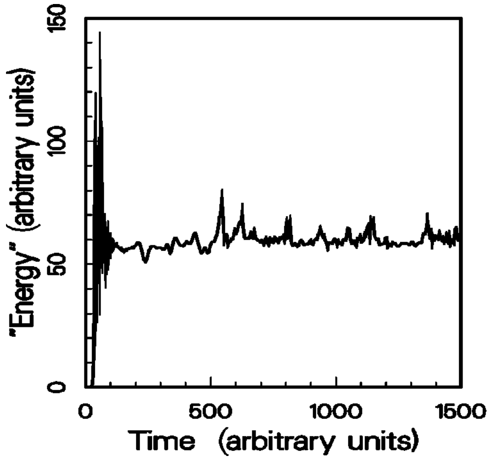

FIG. 1. Evolution of energy illustrating that the solutions of Eq. (96) remain bounded.

$$y=L_a\tilde{y}, \quad t=T_a\tilde{t}, \tag{95}$$

where $N=1/(2L_a^2)$ and $T_a=L_a^3/2$. Dropping the tildes in the notations of variables, the resulting canonical EE is

$$\eta_t+\eta\eta_z-\kappa\eta_{yy}+\nabla^2\eta_z+\epsilon(\eta_{zz}+\nabla^4\eta)=0. \tag{96}$$

The control parameters in this equation are

$$\epsilon=\frac{1}{3}\sqrt{W\delta} \tag{97}$$

and

$$\kappa=\frac{L_a\cot\theta}{3}=\frac{1}{3}\sqrt{\frac{W}{\delta}}\cot\theta. \tag{98}$$

For the case of $\kappa=0$ (e.g., flow down a vertical wall), Eq. (96) becomes essentially the model equation postulated and numerically studied in Ref. [26]. If, in addition, $\varepsilon=\kappa=0$, we have the 2D KdV equation [33], whose 1D ($\partial_y=0$) version is the usual KdV equation. When $\kappa=0$ but $\epsilon\to\infty$ (so that, after an appropriate rescaling, the dispersive term disappears from the equation), Eq. (96) becomes the one obtained by Nepomnyashchy [34], whose 1D version is the Kuramoto-Sivashinsky equation [35,36]. All of these equations are thus limiting cases of EE (96).

Numerical simulations [22] of this equation have shown that its solutions remain bounded for all time (which property always is implicitly *assumed* in global-validity considerations, and therefore must be directly verified *after* the EE has been obtained). For example, Fig. 1 shows the evolution of the surface deviation ''energy'' $\int\eta^2\,dy\,dz$ from an initial small-amplitude ($\eta\sim10^{-2}$) ''white-noise'' condition to a statistically steady state (for $\kappa=0$ and $\epsilon=1/50$). Detailed numerical studies of Eq. (96) will be presented elsewhere.

V. ADDITIONAL DISSIPATIVE TERMS

In this section, we examine the implications of the additional dissipative terms arising from further iterations. We take into account explicitly all the linear dissipative and dispersive terms with derivatives of order four or less by going through the iterative process twice. We also estimate the effect of dissipative terms with derivatives of order six or more on the EE.

We note that, for obtaining the evolution equation, we need only the successive iterates of the normal velocity u. This is because the nonlinear terms involving $w\eta_z$ and $v\eta_y$ in the kinematic condition make smaller contributions to the EE. (Indeed $u\sim w_z+v_y$ and, e.g., $w_z\eta$ generates the same terms as the w_z part of u, but with the extra small factor η.) We have found the next two iterative corrections for the normal velocity, u_2 and u_3, Eqs. (B22) and (B37) in Appendix B of Ref. [25]. [We need u_3 (in addition to u_2) because its dissipative terms are not guaranteed to be much smaller than those of u_2. However, the dissipative terms of u_4 *are* much smaller than those of u_2, so we do not have to consider u_4.] Using these in the kinematic condition (9), we obtain [see Appendix B of Ref. [25] for details]

$$\begin{aligned}\eta_t=&\left[\frac{5}{12}R\eta_z-\frac{4}{15}R\cot\theta\nabla^2\eta+\frac{295}{672}R^2\eta_{zz}\right]_t-4\eta\eta_z-\frac{2}{3}\delta\eta_{zz}+\frac{2}{3}\cot\theta\,\eta_{yy}-2\nabla^2\eta_z-\frac{2}{3}W\nabla^4\eta\\&-\left[\frac{23}{15}R-2\cot\theta\right](\eta\eta_z)_z+\frac{5}{14}R\cot\theta\nabla^2\eta_z-\frac{2}{7}R^2\eta_{zzz}\\&+\frac{6}{5}\cot\theta\nabla^4\eta-\frac{331}{168}R\nabla^2\eta_{zz}-\frac{1\,241\,483}{8\,108\,100}R^3\eta_{zzzz}+\frac{477\,523}{2\,494\,800}R^2\cot\theta\nabla^2\eta_{zz}.\end{aligned} \tag{99}$$

We get rid of time derivatives on the RHS by twice iterating this equation, substituting (for each time derivative on the RHS) the RHS (99) of Eq. (99) itself; the remaining time derivatives on the final RHS are omitted because further iterations would only lead to derivatives of an order higher than 4, with small resulting contributions. [In fact, in the first iteration, when substituting into the (mixed-derivative) terms with the second spatial derivatives, it is enough to retain from the RHS (99) only the terms with *no* time derivatives, and with space-derivatives of the order of 2 only. As for the term with the first spatial derivative, $\propto\eta_{tz}$, the terms without time derivatives and with space derivatives of orders 2 and 3 are substituted into it, and also the term $(5/12)R\eta_{tz}$ itself, with the result $(5/12)^2R^2\eta_{tzz}$. In the second iteration, it is sufficient to only substitute into the latter term, and—since it already contains two spatial differentiations—only the purely spatial second-derivative terms should be substituted into it.] As a result, we obtain the following equation:

$$\eta_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{2}{3}\cot\theta\,\eta_{yy}+2\nabla^2\eta_z+\frac{40}{63}R\delta\eta_{zzz}-\frac{40}{63}R\cot\theta\,\eta_{zyy}+\frac{2}{3}W\nabla^4\eta-\frac{2}{3}\cot\theta\nabla^4\eta+\frac{157}{56}R\nabla^2\eta_{zz}+\frac{8}{45}R\cot^2\theta\nabla^4\eta+\frac{1\,213\,952}{2\,027\,025}R^3\eta_{zzzz}-\frac{138\,904}{155\,925}R^2\cot\theta\nabla^2\eta_{zz}+\left(\frac{16}{5}R-2\cot\theta\right)(\eta\eta_z)_z=0. \quad (100)$$

The 1D ($\partial_y=0$) limit of this equation coincides with the small-amplitude limit of the EE obtained, with the same numerical coefficients, in Ref. [14], but our 2D version is new. [We remark that there are several mistakes in the presentation of steps leading to the final equation, Eq. (27) in Ref. [14]. However, Eq. (27) itself appears to be correct. The same numerical coefficients appeared in an even earlier paper [11] in a linearized 1D context.] The (2D) terms with derivatives of order 3 or less agree with Ref. [15], and those plus the surface-tension (W) term—with Ref. [37]. However, note that some terms of (100) are always negligible. For example, the two dispersive third-derivative terms containing R are clearly smaller than the corresponding second-order dissipative terms (because of the instantaneous VCs $R/L \ll 1$ and $\cot\theta/L \ll 1$), and therefore are negligible in all cases.

As was mentioned above, since we are only interested in situations where persistent nonlinear waves are present, $\delta>0$, we have either $\delta\sim R$ or $\delta\ll R$ [but not $\delta\gg R$; see Eq. (81)]. When $\delta\sim R$, the additional fourth-derivative terms in Eq. (82) are each smaller than the destabilizing second-derivative term (because of $R/L\ll 1$ and/or $1/L^2\ll 1$). If $\max(R,R^3)\ll W$ [see Eq. (92)], those terms are much smaller than the stabilizing *capillary* term, and we return to the GEE (82) with global VCs (93) and (94) (this holds as well even for $\delta\ll R$). But if W is not large enough, so that Eq. (92) is violated and the capillary fourth-derivative term is much smaller than (at least) one of the noncapillary fourth-derivative members, then the destabilizing term cannot be balanced, and the EE leads to the unlimited growth of amplitude. Thus, clearly, at such parameter-space locations, the EE (100) *can* be valid locally, i.e., for a limited time only, but it is *not* globally good.

Consider now (for the rest of this section) the complementary case $\delta\ll R$ (so $R\simeq R_c$), and with W sufficiently small, so that the condition $\max(R,R^3)\ll W$ is violated. We have obtained, except for numerical coefficients, all of the essential terms (which turn out to be linear; see the end of Appendix C of Ref. [25]). Including these terms in Eq. (100), the EE can be written in the general form

$$\eta_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{8}{15}R_c\eta_{yy}+2\eta_{zzz}+\frac{2}{3}W\eta_{zzzz}+\frac{8}{5}R_c(\eta\eta_z)_z+\left[\frac{2581}{1400}R_c-\frac{32}{3\,378\,375}R_c^3\right]\eta_{zzzz}+[\text{sum of terms of type } R_c^{2k+1}\eta^{(2l)}]=0. \quad (101)$$

Here the superscript on η, enclosed in parentheses, refers to the order of the spatial derivative; $l>k\geqslant 0$; $k=0,1,2,\ldots,l=(k+1),(k+2),\ldots$; and every $(k+l)$ is odd. In Eq. (101), we have used the leading Taylor-series approximation putting $R=R_c$ (recall also $\cot\theta\approx 4R_c/5$). Also, $Y\gg Z$ (as a consequence of $\delta\eta_{zz}\gg R_c\eta_{yy}$, which is required for instability to develop). Furthermore, we have not included additional nonlinear or dispersive terms in Eq. (101); the dissipative nonlinear terms can be shown (see Appendix C of Ref. [25]) to be smaller than the leading nonlinear dissipative term $\propto R(\eta\eta_z)_z$, and all the dispersive terms are smaller than the linear one $\nabla^2\eta_z$.

Normally, the coefficients of the terms are $O_M(1)$. However, sometimes a coefficient is $\ll 1$ because of an accidental near cancellation of terms as, e.g., is the case for the term $\propto R_c^3\eta_{zzzz}$ in Eq. (101). Then we say the term is ''degenerate.'' Comparing the additional dissipative terms with the term $\propto R_c^3\eta_{zzzz}$ and taking into account the instantaneous VCs $R/L\ll 1$ and $1/L^2\ll 1$, the only terms which may not be negligible are those of the structure $R_c^{2n-1}\eta^{(2n)}$ ($n=2,3,\ldots$) (even those could have been neglected in comparison with the term $\propto R_c^3\eta_{zzzz}$ were the latter nondegenerate). Hence, the EE can be simplified:

$$\eta_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{8}{15}R_c\eta_{yy}+2\eta_{zzz}+\frac{8}{5}R_c(\eta\eta_z)_z+\left[\frac{2581}{1400}R_c-\frac{32}{3\,378\,375}R_c^3\right]\eta_{zzzz}+\sum_{n=3}^{\infty}c_nR_c^{2n-1}\eta^{(2n)}=0, \quad (102)$$

where c_n are numerical coefficients. Can this equation be valid for all time? For this to be the case, the destabilizing term $(2/3)\delta\eta_{zz}$ has to be balanced by a stabilizing one—by the term $\propto R_c\eta_{zzzz}$ (the other, degenerate fourth-derivative term is clearly destabilizing) or by the first nondegenerate and stabilizing higher-derivative term of Eq. (102), whichever term is dominant.

Suppose first that the fourth-derivative term is the dominant stabilizing one. Then Eq. (102) becomes

$$\eta_t+4\eta\eta_z+\frac{2}{3}\delta\eta_{zz}-\frac{8}{15}R_c\eta_{yy}+2\eta_{zzz}+\frac{8}{5}R_c(\eta\eta_z)_z+\frac{2581}{1400}R_c\eta_{zzzz}=0. \quad (103)$$

The balance of the destabilizing term $\delta\eta_{zz}$ with the stabilizing term $R\eta_{zzzz}$ yields the length scale $L\sim R_c/\delta$. Comparing the dissipative term with the dispersive term, we have $R_c\eta_{zzzz}/\eta_{zzz}\sim R_c/L\ll 1$; the dispersive term is always domi-

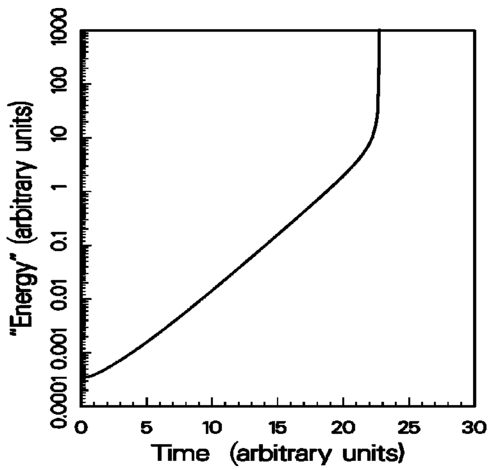

FIG. 2. Evolution of energy indicating that the solutions of Eq. (104) blow up.

nant. Therefore, it is the dispersive term which is to balance the nonlinear one, which yields the characteristic amplitude $\sim 1/L^2$. Using this, it is easy to see that the nonlinear dissipative term $\sim R_c(\eta\eta_z)_z$ is exactly O_M of the linear dissipative terms; therefore, the nonlinear term plays a significant role. It is destabilizing, and numerical simulations indicate that the solutions of Eq. (103) blow up [see Fig. 2, which shows the blowup of energy (in contrast to Fig. 1) governed by the equation rescaled to the canonical form similar to Eq. (96),

$$\eta_t+\eta\eta_z+\eta_{zzz}-\kappa\eta_{yy} +\varepsilon[\eta_{zz}+\eta_{zzzz}+(1120/2581)(\eta\eta_z)_z]=0, \tag{104}$$

with $\kappa=0$ and $\varepsilon=1$; we have observed similar blowup behavior with all the values of ε we tested in the range from 10^{-2} to 10^2]. The thickness of the real film, of course, is bounded uniformly for all time. Hence, Eq. (103) (which *can* be good for a limited time) is *not* globally valid. Physically, we believe that the growth of amplitude will be arrested by viscosity after small length scales develop, which would violate the small R/L VC for the single EE description. So the EE (103) cannot be valid for large times.

The hypothetical case when a higher-derivative term $\propto R_c^{2n-1}\eta^{(2n)}$ is the dominant stabilizing one remains to be considered. [In particular, this implies that $3\leqslant n=m$ (say), $c_n\ll 1$ for $n<m$, $c_m\sim 1$, and c_m has an appropriate sign; $c_m>0$ if m is even and $c_m<0$ if m is odd.] We do not believe this actually happens; such terms are traced back to be due to the inertia terms in the momentum NS equations, the same inertia terms which give the destabilizing second-derivative term of Eq. (102), and it seems unlikely that the same physical cause can be responsible for both stabilization and destabilization. Therefore, we believe the first nondegenerate term will turn out to be *destabilizing*.

Based on the (linear) studies [38] of the Orr-Sommerfeld equation, we have computed the next coefficient, which shows that, albeit stabilizing, the sixth-derivative term is again degenerate: $c_3=-16\,173\,184/1\,718\,663\,821\,875\approx -0.941\times 10^{-5}$. (This is also remarkably close to the fourth-derivative coefficient, $c_2\approx -0.947\times 10^{-5}$. If further c_n were all degenerate too and of the same order of magnitude, the destabilizing fourth-derivative term would be dominant, and hence stabilization and all-time valid EE impossible.).

A special case to consider is when the destabilizing fourth-derivative term (which is clearly much greater than the sixth-derivative one) is nearly cancelled by the stabilizing fourth-derivative term $aR_c\eta^{(4)}$ (where $a\equiv 2581/1400$). As the term $c_3R_c^5\eta^{(6)}$ is stabilizing, one [39] can ask whether there can be an all-time valid EE if $|c_3R_c^5\eta^{(6)}|\gg|aR_c - c_2R_c^3)\eta^{(4)}|$. Our answer to this question is as follows. In this case, $aR_c\eta^{(4)}\sim|c_2|R_c^3\eta^{(4)}\gg|c_3|R_c^5\eta^{(6)}$. Since the dispersive (third-derivative) term must dominate the fourth-derivative term $aR_c\eta^{(4)}$, it dominates even stronger the sixth-derivative term $c_3R_c^5\eta^{(6)}$ [see the discussion following Eq. (103)]. As usual, the length scale is determined by a balance between the dominant linear dissipative terms, $\delta\eta_{zz}\sim|c_3|R_c^5\eta^{(6)}\ll aR_c\eta^{(4)}\sim R_c(\eta\eta_z)_z$ [see the arguments following Eq. (103)]. Thus, the destabilizing nonlinear term is the greatest one; there is no other term that could serve as a counterbalance. We arrive at the conclusion that the hypothetical EE with the sixth-derivative term cannot be valid for all time.

We have not attempted to determine the next coefficient, c_4, because of the large volume of calculations that would be required. Based on what we have said above, we expect the eighth-derivative term to be nondegenerate and destabilizing, and then no single EE can be globally valid under the circumstances. It follows that the GEE (82) *is* the most general one. The validity condition $W\gg R_c^3$ (which would be sufficient even if c_2 were nondegenerate) can be relaxed a bit; it is enough to require that the capillary term dominates the (presumably nondegenerate) eighth-derivative one, $W/L^3 \gg R_c^7/L^8$. (With $L^2\sim W/\delta$ [see Eq. (90)], we get $W^{7/2}/(\delta^{5/2}R_c^7)\gg 1$.) The EE (102) is valid locally only, under the instantaneous VCs (89).

If, however, the first nondegenerate term $\propto R_c^{2n-1}\eta^{(2n)}$ turned out to be stabilizing, the EE (102) would have a domain of global validity, albeit a very limited one. It is straightforward to find the corresponding global VCs. Indeed, the balance between this term and the destabilizing one, $R_c^{2m-1}\eta^{(2m)}\sim\delta\eta_{zz}$, yields the length scale L,

$$L^{2m-2}\sim\frac{R_c^{2m-1}}{\delta}. \tag{105}$$

Using this length scale, the ''modified-R'' VC takes the form

$$\frac{R_c}{L}\sim\left(\frac{\delta}{R_c}\right)^{1/(2m-2)}\ll 1, \tag{106}$$

and the small-slope VC becomes

$$\frac{1}{L^2}\sim\left[\frac{\delta}{R_c{}^{(2m-1)}}\right]^{1/(m-1)}\ll 1. \tag{107}$$

The (relaxed) conditions of dominance are $W\eta^{(4)} \ll R_c^{2m-1}\eta^{(2m)}$ and $R_c\eta^{(4)}\ll R_c^{2m-1}\eta^{(2m)}$, i.e., $W \ll R_c^{2m-1}/L^{2m-4}$ and $R_c^2(R_c/L)^{2m-4}\gg 1$ [which with Eq. (105) are easy to recast in terms of the global parameters only]. The ratio of the nonlinear dissipative term, $\propto R_c(\eta\eta_z)_z$, to the stabilizing one, $\propto R_c^{2m-1}\eta^{(2m)}$, is $A/(R_c/L)^{2m-2}$, which is required to be small—otherwise, we can have the blowup of solutions, similar to the case with the term $\propto R_c\eta^{(4)}$ being dominant. The comparison of dissipative term, $\propto R_c^{2m-1}\eta^{(2m)}$, to the dispersive term, $\propto \eta_{zzz}$, shows that the latter can be greater or smaller than the former depending upon whether $R_c^{2m-1}/L^{2m-3} \ll 1$ or $R_c^{2m-1}/L^{2m-3} \gg 1$.

When $R_c^{2m-1}/L^{2m-3} \ll 1$ (dispersion is large), the characteristic amplitude, obtained by balancing the term $\eta\eta_z$ with the dispersive term, is $1/L^2$ [hence, the small-amplitude VC, $A \ll 1$, coincides with Eq. (107)]. Otherwise, i.e., if $R_c^{2m-1}/L^{2m-3} \gg 1$, the amplitude is determined by balancing the term $\propto \eta\eta_z$ with the dominant stabilizing term, $A \sim (R_c/L)^{2m-1}$. Using the global-parameter estimate of the length scale [Eq. (105)], all of the above conditions are readily reduced to certain global VCs (expressed in terms of global parameters only); we do not write them here, in view of the likely nonreality of the imaginary case of dominance of the term $\propto R_c^{2n-1}\eta^{(2n)}$.

VI. SUMMARY

We have considered flow of a liquid film down an inclined plane. We have posed and studied the following questions: What are the least restrictive parametric conditions for which the wavy film flow can be approximated for all time by a single (local) evolution equation, and what (if any) is the most general form of such an equation?

We have argued that the dissipative-dispersive evolution equation (82) [which we derived by an iterative perturbation method of a multiparametric type] is such a general EE. Any all-time valid EE derived by a single-parameter technique is necessarily nothing else but essentially the general EE in which some terms have been omitted. Also, the domain of validity of such a ‘‘partial’’ EE is necessarily a *subdomain* of the ‘‘umbrella’’ domain of global validity given by Eqs. (93) and (94). [In particular, in such domains the amplitude of waves is necessarily much smaller than the mean film thickness.]

It is clear that any evolution equation which follows from a multiparametric approach (such as the iterative technique we have employed in this paper) can be also obtained with the conventional SP approach. However, the significant advantage of the MP derivation is that it covers at once all possible SP derivations of EEs (the number of which is, in principle, infinite in the SP approach, corresponding to the different choices of the small-parameter powers for the system parameters). Also, comparing the two derivations of even a *particular* EE, the MP derivation is justifiable for much less restrictive domains of the parameter space.

The theory also yields the explicit approximate expressions in terms of film thickness for the pressure and components of velocity [Eqs. (50), (42), (58), and (36)], and thus a complete description of film dynamics.

We have derived an EE (100) containing additional high-derivative terms which can be essential near the threshold of instability, $R \approx R_c$, if the surface tension is sufficiently small. However, this EE (102) is good for a limited time only and cannot be good for all time. [The conditions of such a local (in time) validity are given by Eq. (89).] Under certain parametric conditions, the (numerical) solutions of that EE blow up due to a nonlinear (quadratic, second-derivative) destabilizing term.

The EE (82) is relatively easy for numerical simulations of the 3D waves in the inclined film. We have obtained good agreement with transient states and transitions observed in the physical experiments of Ref. [9]. Under certain parametric conditions for which the dissipative terms of the EE are small, we observed self-organization (from the initial white-noise small-amplitude conditions) of unusual highly ordered patterns of solitonlike structures on the film surface (the pattern consists of two traveling-wave subpatterns which move with different velocities). The studies of the evolution equation (82) will be published elsewhere (see also Refs. [4,22]).

A similar analysis leads to analogous single EE theory of a film flowing down a vertical ‘‘fiber.’’ We believe that similar theories can be useful for a variety of other systems.

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APPENDIX: VALIDITY CONDITIONS

From Eq. (73), the estimate of w_1, in terms of the basic parameters and length and amplitude scales, is

$$w_1 \sim \max\left(\frac{1}{Z^2},\frac{1}{Y^2},\frac{R}{Z},\frac{\cot\theta}{Z},\frac{R}{T},\frac{W}{Z^3},\frac{W}{ZY^2}\right)A. \quad \text{(A1)}$$

Estimating u_1 from Eq. (77), we have

$$u_1 \sim \max\left(\frac{1}{Z^3},\frac{1}{ZY^2},\frac{R}{Z^2},\frac{\cot\theta}{Z^2},\frac{\cot\theta}{Y^2},\frac{R}{TZ},\frac{W}{Z^4},\frac{W}{Z^2Y^2},\frac{W}{Y^4}\right)A. \quad \text{(A2)}$$

In obtaining a solvable equation for the boundary condition on w_{1x} at $x=1$ [Eq. (71)], we have dropped the term $u_{1z}(x=1)$. This implies

$$u_{1z} \ll w_{1x} \quad (x=1). \quad \text{(A3)}$$

Using the O_M estimates for u_1, Eq. (A2), and $w_{1x}(x=1)$, Eq. (71), the above requirement reduces to

$$u_{1z}(x=1) \sim \max\left(\frac{1}{Z^3},\frac{1}{ZY^2},\frac{R}{Z^2},\frac{\cot\theta}{Z^2},\frac{\cot\theta}{Y^2},\frac{R}{TZ},\frac{W}{Z^4},\frac{W}{Y^2Z^2},\frac{W}{Y^4}\right)\left(\frac{A}{Z}\right) \ll w_{1x}(x=1) \sim \frac{A}{Z^2}. \quad \text{(A4)}$$

This again yields the conditions (28), (29), (30), (31) and, in addition, the following conditions:

$$\frac{\cot\theta}{Z} \ll 1, \quad \text{(A5)}$$

$$\left(\frac{Z}{Y}\right)\left(\frac{\cot\theta}{Y}\right)\ll 1, \tag{A6}$$

$$\frac{W}{Z^3}\ll 1, \tag{A7}$$

$$\frac{W}{ZY^2}\ll 1, \tag{A8}$$

and

$$\left(\frac{Z}{Y}\right)\left(\frac{W}{Y^3}\right)\ll 1. \tag{A9}$$

These conditions, along with Eqs. (11) and (28)–(31), form the complete set of validity conditions for the present theory. They are sufficient to justify all the simplifications of the equations. Thus, the complete set of VCs is

$$\max\left[A,\frac{1}{Z^2},\frac{1}{Y^2},\frac{R}{Z},\frac{\cot\theta}{Z},\frac{Z\cot\theta}{Y^2},\frac{R}{T},\frac{W}{Z^3},\frac{W}{Y^2Z},\left(\frac{Z}{Y}\right)\frac{WZ}{Y^4}\right]\ll 1. \tag{A10}$$

Using these conditions, it is easy to see that $w_1\ll w_0$ and $u_1\ll u_0$.

One can estimate the O_M of Rp_1, by using Eq. (B6) of Ref. [25] as

$$Rp_1\sim\max\left(\frac{A}{Z},\frac{1}{Z^3},\frac{1}{ZY^2},\frac{R}{Z^2},\frac{\cot\theta}{Z^2},\frac{\cot\theta}{Y^2},\frac{R}{TZ},\frac{W}{Z^4},\frac{W}{Z^2Y^2},\frac{W}{Y^4}\right)A. \tag{A11}$$

Using the conditions (A10), it is easy to show that $Rp_1/Z \ll Rp_0/Z$ or $p_1\ll p_0$. By using Eq. (B12) of Ref. [25], one can estimate the O_M of v_1 as

$$v_1\sim\max\left[\frac{A}{YZ},\frac{1}{YZ^3},\frac{1}{ZY^3},\frac{R}{YZ^2},\frac{\cot\theta}{YZ^2},\frac{\cot\theta}{Y^3},\frac{R}{TYZ},\frac{W}{YZ^4},\frac{W}{Y^3Z^2},\frac{W}{Y^5},\frac{A\cot\theta}{Y},\frac{AW}{YZ^2},\frac{AW}{Y^3},\frac{R\cot\theta}{ZY},\frac{RW}{YZ^3},\frac{RW}{ZY^3},\frac{R\cot\theta}{TY},\frac{RW}{TY^2Z},\frac{RW}{TY^3}\right]A. \tag{A12}$$

Using the estimate of v_0 [Eq. (59)] and the conditions (A10), it is easy to see that $v_1\ll v_0$. The VCs (A10) guarantee that all the terms involving w_0, u_0, v_0, p_0, w_1, u_1, v_1, and p_1 that were dropped in obtaining solvable ODEs for the same quantities are small in comparison with the terms that were retained.

Estimating the O_M of various terms in Eq. (82), and noting that $\delta\ll R$ or $\delta\sim R$, we find that

$$\frac{A}{T}\sim\max\left[A,\frac{1}{Z^2},\frac{1}{Y^2},\frac{\delta}{Z},\frac{\cot\theta}{Z},\frac{Z\cot\theta}{Y^2},\frac{W}{Z^3},\frac{W}{ZY^2},\frac{WZ}{Y^4}\left(\frac{A}{Z}\right)\right] \tag{A13}$$

and, consequently, taking Eq. (A10) into account,

$$\frac{R}{T}\sim\max\left[A,\frac{1}{Z^2},\frac{1}{Y^2},\frac{\delta}{Z},\frac{\cot\theta}{Z},\frac{\cot\theta Z}{Y^2},\frac{W}{Z^3},\frac{WZ}{Y^4},\frac{W}{ZY^2}\left(\frac{R}{Z}\right)\right]\ll 1. \tag{A14}$$

Hence, the parameter R/T is small as a consequence of the smallness of other parameters in Eq. (A10), and thus can be omitted from there. Also, Eq. (87) yields $\cot\theta/Z < R/Z \ll 1$, so that the parameter $\cot\theta/Z$ can be omitted as well. The somewhat simplified validity conditions are

$$\max\left[A,\frac{1}{Z^2},\frac{1}{Y^2},\frac{R}{Z},\frac{\cot\theta Z}{Y^2},\frac{W}{Z^3},\frac{W}{Y^2Z},\frac{WZ}{Y^4}\right]\ll 1 \tag{A15}$$

(which, in turn, significantly simplify [see Eq. (89)] if $Y\geqslant Z$).

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