ON SATURATION OF RAYLEIGH-TAYLOR INSTABILITY

A. L. FRENKEL AND D. HALPERN
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487

We derive a generic, "template" evolution equation for the thickness of a film in a single- or two-fluid system under the parametric conditions when the film inertia and the adjoining-liquid (if any) disturbances are negligible in the dynamical problem for the film disturbances. For the plane Couette-type flow between two horizontal plates (with the lighter film layer at the bottom) generated by an in-plane circular spinless motion of the upper plate, a small-amplitude evolution equation follows. Its numerical simulations confirm that the growth of any – in general, three-dimensional – disturbance is arrested via nonlinear effects, with the disturbance of thickness remaining small. A physical example of a system whose parameters satisfy the conditions of consistency for the derivation of the approximate evolution equation, is given. The possibility of a non-dripping wet ceiling is discussed and experiments with moving ceilings are suggested.

1. Introduction

As is well known, the gravitational *Rayleigh-Taylor* (RT) instability of fluid layers occurs in various industrial and natural processes. Examples include coating flows with paint or photographic material, the boiling of liquids, inertial confinement fusion experiments in which pellets of deuterium-tritium fuel undergo laser implosion, and certain geophysical processes. Rupture is often undesirable, as in the breakup of the shell containing the fuel before the fuel is fully compressed in inertial confinement fusion (e.g., [1]).

The linear theory of the RT instability is documented in [2], and later developments, including the work on two-layer *plane Couette* (PC) flows of fluids with different densities, are discussed in [3]. In the 1980s and the early 1990s, advances were made in understanding the *nonlinear* development of the longwave RT instability in a viscous film. The possibility of *small-amplitude saturation* of the RT instability, such that the amplitude of the interfacial waves remains *small* as compared to the average film thickness, was indicated in [4]. By using a weakly nonlinear theory, a one-dimensional (1D) evolution equation (EE) of the Kuramoto-Sivashinsky (KS) type was obtained for the undulation of the interface in the case of a *steady linear* profile of the base velocity between two horizontal plates, the *classical* two-fluid PC flow. Experiments of the 3D instability of a RT system with no basic velocity field, a "film on a ceiling" [5], have demonstrated that at large times, drop coalescence is possible leading to dripping for even very thin films.

As for the saturation result of [4], it was obtained for 2D (streamwise) disturbances only, i.e. for an infinitely small fraction of all possible, in general 3D, disturbances. The nonlinear mechanism of the saturation [6] depends crucially on the nonlinear term of the KS equation, which vanishes for any spanwise disturbance; as a result, for the *classical* PC flow, the saturation fails, in general. This consideration leads to the idea that one needs a base flow with time-dependent velocities, rotating through *all horizontal directions*, in order for any direction to be, for some time, almost like the streamwise direction in the classical PC flow. Then, the saturation mechanism can work, even if only part-time, in *every* horizontal direction, thus arresting the growth of an *arbitrary* 3D disturbance; that is, a *genuine* small-amplitude saturation of the RT instability is achieved. The investigation of this idea is the main subject of the present communication. We also discuss the suggestion that a similar horizontal motion of a *wet ceiling* can keep it from dripping.

2. Generalized two-fluid plane Couette flow

Consider a two-fluid system between two horizontal plates which are a distance $h + h_L$ apart, with the upper plate moving in its plane with an assigned horizontal, but otherwise arbitrary, time-dependent velocity $\mathbf{V}_u(t) = U_u(t)\mathbf{i} + W_u(t)\mathbf{k}$ (where $\mathbf{i}$ and $\mathbf{k}$ are the unit vectors in the horizontal directions x and z, respectively) and the bottom plate being fixed. A fluid layer of constant viscosity μ_F and density ρ_F is bounded below by the fixed plate at $y = -h$, and above (at $y = 0$) by another liquid layer (of thickness h_L) which has a density ρ_L and viscosity μ_L. It is natural to look for the flow whose velocity field is also horizontal everywhere but depends only on the *vertical* coordinate y (and time), with the pressure depending only on y. We denote $\mathbf{V}_F(y,t)$ and $\mathbf{V}_L(y,t)$ the velocity fields in the lower and upper fluids, respectively. Let their components be $[U_j,W_j]{:=}\mathbf{V}_j \cdot [\mathbf{i}, \mathbf{k}]$ where $j = F$ for the film and $j = L$ for the upper liquid (the symbols := or =: indicate the definition of the quantity appearing next to the colon). The Navier-Stokes (NS) equations take the simple form of the familiar "diffusion" equation:

$$\rho_j(U_j)_t = \mu_j(U_j)_{yy}, \ \ \rho_j(W_j)_t = \mu_j(W_j)_{yy} \tag{1}$$

The boundary conditions (BCs) are as follows: (i) the no-slip at the plates, $[U_F,W_F](y=-h)=0$ and $[U_L,W_L](y=h_L)=[U_u,W_u](t)$, and (ii) the conditions of continuity of velocities and shear stresses at the film interface, $[U_F,W_F](0,t)$=$[U_L,W_L](0,t)$ and $\mu_F[U_F,W_F]_y(0,t)$=$\mu_L[U_L,W_L)](0,t)$, respectively.

The unique solution of this problem with arbitrary initial conditions is known to exist for any (reasonably smooth) initial conditions (and can be always obtained in series form). It seems natural to call these flows "the generalized two-fluid PC flows", the classical PC flow being a particular case, the one of time-independent velocities.

3. Film disturbances and "template" evolution equation

We focus on the *film* case of the generic PC flows with $h \ll h_L$. We denote u, v, w, p the disturbances of the x-, y-, z-components of velocity and of pressure, respectively, in the *film*. Let us write down the dynamic equations for the disturbances in a generic form, without specifying the base velocity profile. Using the combination of the (long-wave) lubrication and Stokes approximations, the simplified (x,z)- momentum, y-momentum, and continuity equations are, respectively,

$$[u,w]_{yy} = [p_x,p_z]/\mu, \qquad p_y = 0, \qquad v_y = -u_x - w_z \tag{2}$$

(we use the notation μ interchangeably with μ_F). At the fixed lower plate, the no-slip BCs are $[u,w](y=-h)=0$ and $v(y=-h)=0$. At the film-liquid interface, $y =: \eta(x,t)$, the balances of normal and tangential stresses are approximated—assuming, in particular, that the *disturbances in the top liquid are negligible*—in the following simplified form:

$$p(y=\eta) = -\sigma\nabla^2\eta - \delta\eta =: \Pi(x,z,t) \tag{3}$$

$$[u,w]_y(y=\eta) = [mU_L - U_F, mW_L - W_F]_y(y=\eta) =: [s,q] \tag{4}$$

Here σ is the interfacial tension; $\nabla := (\mathbf{i}\partial_x+\mathbf{k}\partial_z)$; $\delta := g(\rho_L$-$\rho_F)$ where g is the gravity acceleration; and $m := \mu_L/\mu_F$, the viscosity ratio.

The solution of this problem for the film disturbances is as follows:

$$p(x,y,z,t) = \Pi(x,z,t); \ \ v = -\frac{Y^3}{6\mu}\nabla^2\Pi + \frac{Y^2}{2}\nabla\cdot(s\mathbf{i}+q\mathbf{k}+\frac{1}{\mu}H\nabla\Pi) \tag{5}$$

$$[u,w] = [s,q]Y + [\Pi_x,\Pi_z][Y^2/(2\mu) - H] \tag{6}$$

where $Y{:=}y{+}h$ and $H{:=}h{+}\eta$. Substituting these expressions into the equation of mass conservation, $H_t + (\int_0^H [U_F + u]dY)_x + (\int_0^H [W_F + w]dY)_z = 0$ (or, alternatively, into the well-known kinematic BC), we arrive at

$$H_t + DH_x + EH_z - \nabla \cdot \left(H^3 \nabla \Pi\right) / (3\mu_F) = 0 \tag{7}$$

Here the second and third ("advective") terms depend only on the base velocities, since $[D, E] := \left\{[U_F, W_F] + [S, Q]H + [S, Q]_y H^2/2\right\}_{y=\eta}$ where $[S, Q](y, t) := [mU_L \text{ - } U_F,\, mW_L \text{ - } W_F]_y$; on the contrary, the last term is completely independent of the base velocities [see Eq. (3)]. In the 2D case when $\partial/\partial z = 0$ and $W = 0$, we have $E = 0$, which leads to a 1D equation, $H_t + DH_x - [H^3 \Pi_x]_x/(3\mu_F) = 0$. [After the solution $H(x, t)$ or $H(x, z, t)$ of an EE is substituted back into Eqs. (5-6) for the velocities and the pressure, one arrives at the complete solution of the disturbance problem.]

The (large-amplitude) EE (7) has been obtained here without specifying either the base velocities $U_j(y, t)$, $W_j(y, t)$, or the interfacial pressure disturbance Π. So it is a "template" equation: it can be used to obtain specific EEs for different film systems, with the appropriate changes in the last term of Π [see Eq. (3)] accounting for different destabilizing factors. For example, for (in general, a core-annular flow of) a film in a capillary (when the gravity effects are negligible), the destabilization comes from the transverse (azimuthal) component of the interfacial curvature, and correspondingly the term $[-\delta\eta]$ in Π is replaced by $[-(\sigma/a^2)\eta]$, where a is the radius of the capillary (for a concrete example, see the last paragraph of Section 6).

With $m{=}0$, the template EE is good for *single*-film systems. For two-fluid systems with *no base flow*, the template EE is applicable with $D{=}E{=}0$. Thus, the template EE (7) appears to be remarkably versatile. (However, it does not cover the flows where inertia is essential, such as large-amplitude regimes of wavy, inertia-destabilized, falling films.) Moreover, it admits further generalizations taking into account the viscous (elongational-stress) and inertial NS terms [similar to [7, 8]]; these results, because of space constraints, will be published elsewhere.

By using the template evolution equations, new concrete EEs are easily obtained. Also, a template EE shows connections between various known specific EEs, and avoids unnecessary repetitions in their derivation.

4. Why the earlier saturation results are inadequate

For the classical PC flow with equal viscosities, in the reference frame of the interface, we have a linear time-independent $U_F(y) = yU_0/h$ [so that $U_F(-h) = -U_0$], $S{=}0$, and hence $D{=}U_0\eta/h$ (while $E{=}0$ since $W{=}0$). Then, assuming $\eta \ll h$, Eq. (7) [with Π of (3)] yields a (2D) EE which is written [when appropriately rescaled, with units similar to (9) below] in the canonical form $\eta_t+\eta\eta_x+(\eta_{xx}+\eta_{zz})+(\eta_{xxxx}+2\eta_{xxzz}+\eta_{zzzz})=0$. The KS equation derived in [4] is a particular 1D case of this EE with $\partial_z=0$. Therefore, their conclusion that the (1D) thickness disturbance η remains small forever, i.e. the nonlinear saturation of the instability takes place, is justified only for the (2D) streamwise disturbances of the flow. Allowing for the general, 3D disturbances, for the case when the disturbance is x-independent ($\eta_x = 0$), it is easy to see that the above 2D equation reduces to a *linear* EE which has growing normal-mode solutions, e.g. $\eta \propto e^{k^2 t}e^{ikz}$ for $k \ll 1$ (so that η_{zzzz} is negligible). Thus, there is no small-amplitude saturation for the disturbances with $\eta(t = 0) = f(z)$, for any $f(z)$. (In fact, the same holds for a *generic 2D* disturbance.)

The physical reason for this breakdown of saturation is as follows: For such spanwise disturbances, the nonlinear, streamwise term of the KS equation, which plays a crucial role in the mechanism of saturation [6], is always zero. This consideration leads to the idea that one needs a (time-dependent) base flow whose velocities rotate through *all* horizontal directions—so that any direction, for some time, is almost like the streamwise direction in the classical (time-independent) PC flow. Then the saturation mechanism can work, even if only part-time, for every direction. Such a multidirectional (generalized PC) flow can be obtained as a superposition of two "unilinear", each sinusoidally oscillating, flows in two perpendicular horizontal directions. It turns out that for such a unilinear oscillatory flow, the base velocity profile is found in a *closed* form (given immediately below).

5. Small-amplitude EEs for certain general PC flows

As can be verified by direct substitution, a solution of the first of Eqs. (1) with, for simplicity, $\mu_F=\mu_L$, is

$$U_j(y,t) = \frac{U_0}{h}\Re[e^{i\Omega t} i\{\frac{\sin k_j y}{k_j} + \frac{\tan k_F h \cos k_j y}{k_F}\}] \quad (j=F,\ L) \tag{8}$$

where $k_j:=(\rho_j\Omega/\mu)^{1/2}e^{i3\pi/4}$. These profiles satisfy the required BCs; in particular, $\frac{\partial U_L}{\partial y}(0,t)=\frac{\partial U_F}{\partial y}(0,t)$, the common value being $\frac{U_0}{h}\cos\Omega t$—which shows the meaning of the amplitude constant U_0. Its value is determined by the no-slip condition at the top plate (oscillating with frequency Ω). This amplitude constant U_0 (rather than the maximum speed of the oscillating upper plate, which is a monotonic function of U_0) is convenient to use, together with the frequency Ω, to parametrize this (two-parameter) family of solutions. Each solution has the type of a "Stokes layer" [see, e.g., [9]]: the instantaneous profile of the (unilinear) velocity field is oscillatory in y with the amplitude exponentially decaying downward. The characteristic lengthscales of both the spatial *oscillations* and the *decay* are of the magnitude-order of (for short, $\sim$) $|k_j|^{-1}$ Therefore, if $|hk_j| \ll 1$ (which is the case if Ω is sufficiently small), the instantaneous base velocity profile in the film is practically linear in y.

We can easily conceive of a multidirectional flow as a superposition of such unilinear solutions: since the problems for U_j and W_j are completely decoupled, we have a solution of the same form as (8) for the other velocity component, W_j. The latter, in general, may have a different amplitude constant, W_0 not equal to U_0, and a different phase, $\Omega t \to \Omega t + \phi$—as determined by the motion of the upper plate which generates the PC flow; in particular, for the *circular,* but spinless, motion of the plate, (i) the amplitude constants *are* equal and (ii) $|\phi| = \pi/2$.

We obtain a small-amplitude (2D) EE by substituting into the template EE (7) the base velocity of the multidirectional (spinless) circular PC flow with $U_j(y,t)=\frac{U_0}{h}\Re[e^{i\Omega t} i\psi_j(y)]$ and $W_j(y,t)=\frac{U_0}{h}\Re[e^{i(\Omega t-\pi/2)} i\psi_j(y)]$ [where $\psi_j(y)$ are the factors appearing inside the braces in Eqs. (8)] and assuming that (i) the characteristic lengthscale of the basic flow is large, (ii) $|k_F|h \ll 1$ (we also assume $\rho_F \sim \rho_L$) and (iii) the disturbance remains small-amplitude, $\eta \ll h$. We (i) make the coordinate transformation $[\widetilde{x}, \widetilde{z}] = [x, z] - \int^t [U_j, W_j](0, \tau) d\tau$ (corresponding to the reference frame of the basic interface); (ii) use the leading terms of the Taylor series about $\eta = 0$ in (7); and (iii) rescale the EE to a canonical form by letting $\bar{x} = x/\Lambda$, $\eta = A\bar{\eta}$, $\bar{t} = t/T$ and $\bar{\Omega} = \Omega T$, (where the tilde has been dropped) with

$$\Lambda := (\sigma/\delta)^{1/2}, \quad T := 3\mu\sigma/(h^3\delta^2), \quad A = h^4\delta^{3/2}/(3\mu U_0 \sigma^{1/2}) \qquad (9)$$

As a result, we obtain the following (dimensionless) EE for η:

$$\eta_t + \eta(\eta_x \cos\Omega t + \eta_z \sin\Omega t) + \nabla^2\eta + \nabla^4\eta = 0 \qquad (10)$$

(where the bar over the variables has been dropped). We note that essentially the *same* EE is obtained when the upper plate is fixed and the lower one moves. [This is in contrast to the case [10] of a different (Yih) instability]. Also, we get the same EE when *both* plates oscillate, unilinearly in perpendicular directions; etc..

For the 1D case when $\partial_z = 0$ in (10), we return to the (1D) EE first obtained in [11] [see also [12, 13]]. The 1D equation is clearly of the KS type but the nonlinear term has an oscillatory factor $\cos\Omega t$. We note that Λ, T and A above are the time-asymptotic characteristic amplitude, lengthscale and timescale, respectively, which render all the terms of that "oscillatory KS" (OKS) equation to be pairwise balanced when the nonlinear coefficient is near its maximum value, $\cos\Omega t \approx 1$.

We have simulated the EE (10) with periodic BCs on large domains—of size $2\pi q$ where $q \gg 1$—using a pseudospectral Fourier method [7]. For such large domains, we can expect that solutions [whose characteristic lengthscale is ~ 1] do not significantly depend on the specific type of BCs (except, possibly, for small regions near the boundaries). Also, the further increase of the domain has no significant influence on the character of the solutions. The initial disturbances we employ are, typically, random (e.g., white-noise) and small-amplitude.

For the canonical KS equation, $\eta_t + \eta\eta_x + \eta_{xx} + \eta_{xxxx} = 0$, it was argued [4] that the (infinitely-dimensional) dynamical system essentially forgets any initial conditions and approaches an attractor. In the time-asymptotic regime, there is a pairwise balance of terms in the equation. This leads to the characteristic lengthscale of the wave pattern, its time scale, and the wave amplitude being all ~ 1. Numerical simulations of the KS equation on large intervals agreed with this conclusion. They also revealed chaotic wave patterns suggesting a *strange* attractor in the phase space.

If the frequency Ω in the 1D OKS equation is $\ll 1$, that is much less than the growth rate of the most unstable mode (which is easily seen from the associated linearized EE to be ~ 1), the change of the oscillatory coefficient in the OKS equation is slow, so the EE can be considered as a "nearly constant-coefficient"—containing a variable parameter whose change is ("adiabatically") slow. Then the pairwise balance of terms predicts an amplitude that adiabatically grows following the decrease of the oscillatory coefficient, and similarly falls when the coefficient increases. These considerations (Halpern & Frenkel, 1999) lead to the conclusion that the peak amplitude is $\approx 1/\Omega$ (for small frequencies)—which is confirmed by our numerical simulations. It is natural to expect the same magnitude of the amplitude, $\eta_{\max} \sim 1/\Omega$, for the 2D equation (10), and the numerical experiment confirms this expectation as well: Figure 1 shows that the (instantaneous) maximum (over the computational domain) of the thickness disturbance, after an initial growth, saturates and fluctuates at the predicted magnitude. This means the *small-amplitude saturation* of the RT instability—if the amplitude is small as compared to the average film thickness, $A/\Omega \ll h$. [This (small-amplitude) saturation is a *genuine* one: it holds for an *arbitrary* initial disturbance.]

In view of the "streamwise rivulet" character of the RT instability, one expects for the circular (spinless) PC flow to have waves with their crests and troughs parallel to the instantaneous direction of the base velocity, and so continuously rotating. This expectation is borne out by numerical simulations: Figure 2 shows the rotating pattern of waves, over a quarter of a cycle. Also, the wavelength is ~ 1 and the amplitude is $\sim 1/\Omega$, as was expected.

Our results (not shown here) of simulations for *elliptical*-shear flows (i.e., those with *non-equal* amplitudes U_0 and W_0) and circular-shear flows on

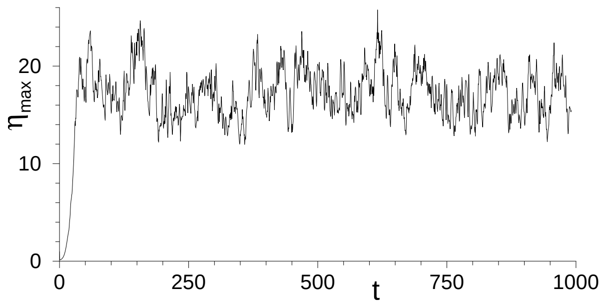

Figure 1. Maximum disturbance ($\eta_{\max}$) of the thickness of a film versus time (t): history of growth and saturation. ($\Omega = 0.025$)

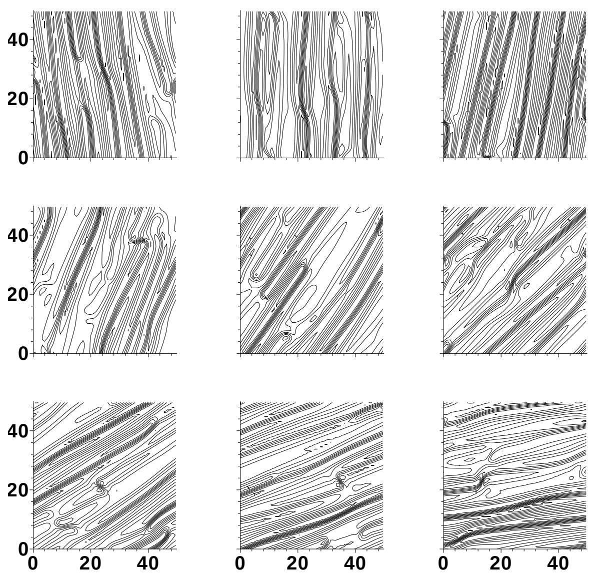

Figure 2. Series of snapshots of film surface (contour plots of constant thickness, with the snapshot time increasing from left to right and from top to bottom of the panel) showing a rotating pattern of waves for a quarter of a cycle.

rectangular *non-square* domains testify to the robustness of the saturation phenomenon.

The above results depend on the EE which was derived by discarding many terms in the NS problem and making certain other assumptions, such as the smallness of the solution amplitude. For consistency, it is required that the neglected terms be much smaller than those retained in the equations for disturbances. Since we have explicit expressions for the velocities and pressure in terms of η and its derivatives [see Eqs. (5-6)], all the members can now be estimated in terms of the basic parameters, with $\eta \sim A/\Omega$, $\partial_x \sim 1/\Lambda$, $\partial_t \sim 1/T$ [where A, Λ, and T are defined by (9)], $\partial/\partial y \sim 1/h$, $\partial^2/\partial y^2 \sim 1/h^2$, etc. (also, since $\rho_L \sim \rho_F$, we use the notation ρ for their common order of magnitude). Neglecting the x-derivative of the velocities in the viscous terms as compared to the y-derivative implies $\Lambda \gg h$, i.e. the parametric constraint $h \ll (\sigma/\delta)^{\frac{1}{2}}$. In order for the adiabatic approximation to be good, one must require $\Omega \ll 1$, or, since $\Omega = \Omega^* T$, where Ω^* is the dimensional frequency, $\Omega^* \ll \frac{h^3\delta^2}{\mu\sigma}$. Other such requirements, which give additional parametric constraints, are as follows: (i) η is small as compared to h; (ii) the basic-flow lengthscale is small, $|kh| \ll 1$; (iii) the inertial terms in the NS equations are negligible; and (iv) the viscous terms in the y-momentum equation [see Eq. (2)] are negligible. All the results can be summarized as the following constraints on U_0 and Ω^* (in terms of the other, "static" parameters):

$$U_0 \ll \frac{\sigma^{1/2}\mu}{h^2\delta^{1/2}\rho}, \quad \Omega^* \ll \min\left(\frac{h^3\delta^2}{\mu\sigma}, \frac{\mu}{\rho h^2}\right), \quad \frac{h^6\delta^{7/2}}{\mu^2\sigma^{3/2}} \ll \Omega^* U_0 \ll \frac{\sigma^{1/2} g}{\delta^{1/2} h}$$

[The other two independent constraints mentioned above, $h \ll h_L$ and $h \ll (\sigma/\delta)^{\frac{1}{2}}$, do not involve either U_0 or Ω^*]. (One can note that, as a consequence of the parametric constraints, Ω^* is bounded away from zero by a quantity *independent* of U_0.)

As a physical example, the system with the following values of parameters (in CGS units) satisfies all the constraints: $\rho_F = 1.0$, $\rho_L = 1.1$, $\mu = 1$, $\sigma = 100$, $h_L = 1$, $h = 0.1$, $U_0 \sim 3$, (so that $\delta \sim 100$, $\Lambda \sim 1$, $T \sim 10$), and $\Omega \sim 0.1$.

The difference of viscosities (neglected above) can play a role via the interfacial contributions of the upper-liquid disturbances—which can be included and analyzed as in [14]. The result is that they are negligible if $|\mu_L - \mu_F| U_0/[(\rho_L - \rho_F)h^2] \ll 1$—in other words, when the RT instability dominates the Yih instability.

6. Ultimate RT instability: film on a ceiling

It is interesting to consider a film spread on a (horizontal) ceiling, the latter being allowed to execute in-plane translational (self-parallel) motions. There is an equilibrium flat-film state (in which the weight of the film is supported by the pressure of the ambient passive atmosphere), but it is *unstable*; this can be considered as, in some sense, the *ultimate* case of the

Rayleigh-Taylor instability. For the *motionless* ceiling, experiments of Fermigier *et al.* [5] showed that the RT instability leads to dripping. In view of our saturation results, it is natural to pose the following question: Can such a dripping be avoided by putting the ceiling in the circular translational (spinless) motion? As was noted above, the template EE holds for such single-fluid cases, $m = 0$. The base velocity (in the laboratory frame) for the unilinear oscillatory flow is $U = U_0\Re(e^{i\Omega t}\cosh ky)$. This gives $D = U_0\Re[e^{i\Omega t}\{\cosh k\eta - k(\sinh k\eta)H - k^2(\cosh k\eta)H^2/2\}]$. Assuming $|k\eta| \ll 1$ and expanding the hyperbolic functions, yields the following small-amplitude evolution equation (in an oscillating reference frame): $\eta_t + 2(\Omega h U_0/\nu)\eta\eta_x \sin\Omega t + h^3(\rho g \eta_{xx} + \sigma\eta_{xxxx})/(3\mu) = 0$. It is clear that for the case of the circular-shear flow generated by the superposition of the perpendicular oscillatory motions with the appropriate phase shift, the small-amplitude 2D evolution equation is

$$\eta_t + 2\Omega h U_0 \eta(\eta_x \sin\Omega t + \eta_z \cos\Omega t)/\nu + h^3(\rho g\nabla^2\eta + \sigma\nabla^4\eta)/(3\mu) = 0$$

However, because of the new factor (proportional to Ω) in the coefficient of the nonlinear term, the consistency conditions are modified. In particular, the requirement that the frequency be much smaller than the growth rate implies $\Omega \ll h^3\rho^2 g^2/(\mu\sigma)$; the requirement that inertia be negligible becomes $U_0 \ll \sigma^{1/2}\mu/(h^2\rho^{3/2}g^{1/2})$; and the requirement that the amplitude be small reads $\rho^{5/2}g^{7/2}h^4/(\mu\sigma^{3/2}) \ll U_0\Omega^2$. Unfortunately, it is easy to see that the first two constraints imply an inequality that is exactly *opposite* to the third constraint. Thus, there is no system which would satisfy the full set of constraints in this case. (This is a stark example of the importance of checking the consistency of the derivation, for every EE.) Therefore, the *small-amplitude* saturation for the film on the ceiling seems unlikely.

This, however does not preclude the possibility that the wet ceiling still can be saved from dripping by the plate motion described above, albeit with the wave amplitudes being "large". The corresponding EE, readily obtained from the template EE with the same base velocity as above (and with $|kh| \ll 1$), is

$$H_t + \Omega U_0 H^2(H_x \sin\Omega t + H_z \cos\Omega t)/\nu + \nabla\cdot[H^3\nabla(\rho g H + \sigma\nabla^2 H)]/(3\mu) = 0$$

The dripping for a sufficiently thin film on the steady ceiling occurs only because of the coalescence of drops [5]. If there is no coalescence of waves for the "film-on-ceiling EE", the dripping can be avoided. It remains to be seen if this is the case; fortunately, the required simulations are similar to [15] [see also [16] and references therein]. Of course, it would be even more interesting to study this question *experimentally*, e.g. by putting into the (in-plane, circular, spinless) motion the "ceiling" which was kept steady in previous experiments [5].

As a related example of the versatility of the template EE (discussed in Section 3), it is immediately clear that certain EEs can be written at once for a *film on* (the outer or the inner side of) a *cylinder* which executes axial *oscillations* (with gravity being negligible): As was mentioned above, one simply makes the substitution $\rho g \to \sigma/a^2$ (where a is the radius of the cylinder) in the EEs of a film on a ceiling (also taking into account that the basic velocity is unilinear). This results in the (2D) evolution equation $H_t + (\Omega U_0/\nu)H^2 H_x \sin\Omega t + \sigma\nabla\cdot[H^3\nabla(H/a^2 + \nabla^2 H)/(3\mu)] = 0$ [where $\nabla = (\partial_x, a^{-1}\partial_\theta)$, θ being the azimuthal angle of the cylindrical coordinates] for the *large*-amplitude regimes [and the OKS-type EE $\eta_t + 2(\Omega h U_0/\nu)\eta\eta_x \sin\Omega t + h^3\sigma(a^{-2}\nabla^2\eta + \nabla^4\eta)/(3\mu) = 0$ for the *small*-amplitude regimes] of flow.

7. Summary

We have obtained a two-dimensional "template" evolution equation: many concrete evolution equations for the film thickness in two-fluid and single-fluid systems follow from it by the simple substitution of the specific base velocities and the interfacial pressure expressions. Hence, we have suggested generalized plane Couette flows which, by rotating though all horizontal directions, afford the small-amplitude saturation of the Rayleigh-Taylor instability for all possible disturbances of the flow. The numerical simulations of the pertinent evolution equation testify to the saturation of the instability and reveal interesting rotating waves in large-time regimes.

We also considered a film spread on a ceiling as a case of the Rayleigh-Taylor instability. Although the small-amplitude evolution equation is readily obtained, the conditions of consistency of the derivation cannot be satisfied. This makes it unlikely that the *small*-amplitude saturation can occur, but does not preclude the possibility for the dripping to be avoided by the circular translational motion of the ceiling—with the wave amplitudes being *as large* as the average thickness of the film. This possibility will be investigated in the near future via numerical simulations of the large-amplitude equation which approximates the evolution of the wavy film. The corresponding laboratory experiment appears to be feasible as well.

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