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LARGE-SCALE INSTABILITY OF GENERALIZED OSCILLATING KOLMOGOROV FLOWS*

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Abstract. The stability of an incompressible unidirectional flow that depends periodically, but otherwise arbitrarily, on a transverse coordinate and on time is considered. An iterative solution of an infinite-dimensional eigenvalue problem is constructed by a rigorous perturbation method. The critical Reynolds number R_c and the critical direction for which the large-scale "eddy viscosity" is minimum (and equal to zero) are determined by a system of two algebraic equations. For both time-independent and time-dependent cases, it turns out that the fastest-growing critical disturbances generally do not have the same transverse periodicity as that of basic flow. In the limit of large frequencies of oscillation, stability is essentially determined by the time-averaged flow. When the latter vanishes, the flow is absolutely stable for sufficiently large frequencies.

Key words. incompressible Newtonian fluid, space-time periodic flows, linear stability, eddy viscosity

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1. Introduction.

Stability of periodic flows of incompressible fluids is currently a topic of active research interest. It is an interesting problem from the viewpoint of the general theory of hydrodynamic stability, and in addition, it has some significant applications.

Notably, such unstable periodic flows can lead to insights for such phenomena of hydrodynamic turbulence as (i) the inverse cascade of energy to—and the negative eddy viscosity [1, 2, 3] at—large scales of motion and (ii) the self-organization of large-scale coherent structures (for an alternative, statistical approach, see, e.g., [4, 5, 6]). Another possible application (as mentioned, e.g., in [7]) is the stability of finite-amplitude waves. Also, certain spatially periodic flows may model large-scale zonal currents in planetary atmospheres, such as those of Earth and Jupiter [8, 9]. Starting with [10], the periodic flows have been studied experimentally as well (by using electrically conducting fluids and external magnetic fields which result in spatially periodic Lorentz forces). Thus, theoretical predictions can be verified in the laboratory.

In their pioneering work [11], Meshalkin and Sinai studied the linear stability of the simplest unbounded periodic flow (suggested to them by Kolmogorov). In an appropriate Cartesian system of coordinates, the velocity field of this "Kolmogorov flow" is in all places parallel to one coordinate axis (in other words, the flow is unidirectional), and this single nonzero component of velocity depends on just one of the other two coordinates (this dependence being simply the sine function).

Meshalkin and Sinai [11] used suitable normal modes to reduce the question of the stability of this flow to an algebraic eigenvalue problem. The components of

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each infinite-dimensional eigenvector are essentially the Fourier coefficients of the corresponding normal mode. The real part of the eigenvalue corresponding to the given mode, the growth rate of the mode, determines its stability properties: the flow is unstable to that mode when its growth rate is positive, and stable otherwise. The normal modes are of the type of the well-known Bloch wavefunctions of the electron in the periodic field of a crystal [12]; each mode has associated with it a certain "quasi wavevector" (similar to the quasi momentum of the Bloch wavefunction). The curve of marginal stability, which gives the Reynolds number corresponding to the marginal stability as a function of the quasi wavenumber (q-wavenumber for short), was found in [11] (however, only those modes whose q-wavevectors were parallel to the basic velocity were considered). The minimum point on this curve has as its coordinates (i) the so-called critical value of the Reynolds number and (ii) the critical q-wavevector, corresponding to the mode to which the flow loses stability as the Reynolds number is raised just over its critical value.

In [11], the critical q-wavenumber turned out to be exactly zero; such a situation is called large-scale instability. However, for other basic flows the critical q-wavevector can be nonzero, e.g., as was established by Takaoka (see [13, 14, 15]) for certain triangular-eddy flows.

When the instability *is* large-scale, the dispersion relation, i.e., the dependence of the growth rate on the q-wavenumber, which gives the most detailed knowledge of stability properties, can also be obtained in the asymptotic limit of small wavenumbers by the multiscale perturbation technique. In fact, most of the work in the area of large-scale instability used this method, which operates with differential equations in contrast to the purely algebraic equations of the eigenvalue approach [13, 14, 15].

A number of results were obtained for various *special* periodic flows. That clearly leaves questionable which of these results will hold for wider classes of flows. Also, until not too long ago, only steady flows were considered. However, it was argued [16] that flows periodic in *both* space and time are more appropriate for turbulence modeling. By employing the multiscale considerations, Hefer and Yakhot [16] obtained the stability properties of a few simple, special space-time periodic (STP) flows.

A question of generalizing the *algebraic* method [11] based on fundamental modes to the case of STP flows was considered in [17]. The fundamental solutions here are *not* normal modes (the dependence on time and on space cannot even be separated into different factors). Rather, each fundamental mode—termed a quasi-normal mode in [17]—is the product of an STP function (of the same space-time periodicity as the basic flow) and two exponential factors, one containing the q-wavenumber for space dependence and the other containing the quasi frequency and growth-rate for time dependence. Expanding the periodic part into the Fourier series, one obtains again an infinite-dimensional (discrete) algebraic eigenvalue problem, whose eigenvalue dependence on the q-wavenumbers is the dispersion relation. In general, the eigenvalues are found numerically (by truncating the infinite matrix); however, in certain cases analytical asymptotic solutions are possible as well, e.g., for the small q-wavenumbers.

On the other hand, the multiscale differential-operator method [18] led to a formally compact general expression for the tensor of large-scale eddy viscosity. (This tensor corresponds to the coefficients of the asymptotic quadratic dependence of the growth rate on the small q-wavenumbers for the large-scale instability.) However, that expression involves inversions of complicated differential operators, which in general requires quite demanding numerical computation [19]. Essentially, analytical solution is possible only for certain special classes of flows. In particular, Dubrulle and Frisch [18] found the eddy viscosity for the *generalized* (time-independent) Kolmogorov flow, in which the velocity is given by a periodic, but otherwise arbitrary, spatial function. Remarkably, it turned out that the critical wavevector can be nonparallel to the basic velocity. This occurrence of "nontransverse" [18] instability is contrary to what was assumed in some earlier papers on the basis of certain known simpler cases, such as the original (sinusoidal) Kolmogorov flow [11].

In the present paper, we consider *oscillating* generalized Kolmogorov flows [17]. These are unidirectional flows such that the (single) nontrivial component of velocity is a product of two factors: an arbitrary periodic spatial function and an arbitrary periodic dependence on *time*. We use the algebraic eigenvalue approach of [17]. Considering the small wavevector magnitude as a perturbation parameter, we construct an iteration sequence for the eigensolution. Remarkably, one can *rigorously* prove its convergence (by means of a contractive mapping theorem), without any truncation of the infinite-dimensional problem (while the use of the perturbation series in [18] was merely formal).

In section 3, we demonstrate this for the simpler, time-independent case (whose eigenvalue problem is introduced in section 2). It is shown in section 4 that the resulting solution completely agrees with the critical parameters obtained in [18], including the generally oblique character of instability. In section 5, we find some general formulas for the eddy viscosity of the *time-dependent* flow (for which no concrete results were available in the literature). The application of those formulas is demonstrated for several specific examples of oscillating Kolmogorov flows in section 6. Finally, section 7 summarizes the paper. Some considerations of a more technical character are relegated to the appendices.

2. Infinite-dimensional eigenvalue problem. For any two-dimensional viscous incompressible flow (with the vectors of the velocity field confined to the (x, y)-plane of the Cartesian coordinate system), the nondimensional Navier–Stokes equation in terms of the stream function $\Phi(x, y, t)$ (where t is time) is

$$\frac{\partial \Delta\Phi}{\partial t} + \frac{\partial(\Delta\Phi, \Phi)}{\partial(x, y)} - \frac{1}{R}\Delta^2\Phi = G, \tag{2.1}$$

where R is the Reynolds number,

$$\frac{\partial(f, g)}{\partial(x, y)} = \frac{\partial f}{\partial x}\,\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\,\frac{\partial g}{\partial x}$$

is the Jacobian of f and g,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplacian, and G is the external forcing. By definition, the relations between velocity $\boldsymbol{v}$ and stream function are $v_x = \frac{\partial \Phi}{\partial y}$ and $v_y = -\frac{\partial \Phi}{\partial x}$.

The equation governing an infinitesimal disturbance $\Psi(x, y, t)$ (superimposed upon the basic flow Φ) follows by linearizing (2.1):

$$\frac{\partial \Delta\Psi}{\partial t} + \frac{\partial(\Delta\Phi, \Psi)}{\partial(x, y)} + \frac{\partial(\Delta\Psi, \Phi)}{\partial(x, y)} - \frac{1}{R}\Delta^2\Psi = 0. \tag{2.2}$$

In this section, we assume the basic flow Φ to be steady, independent of x, periodic in y with period 2π, and of zero mean value:

$$\Phi(x,y,t) = f(y) = \sum_{m=-\infty}^{\infty} b_m \, \mathrm{e}^{\mathrm{i}my} \qquad (b_0 = 0, b_{-m} = b_m^*). \tag{2.3}$$

Then, for any initial disturbance, the solution of (2.2) is a linear superposition (over different "quasi wavenumbers" λ and μ) of the "Bloch components" (discussed in [17]) of the form

$$\Psi(x,y,t) = \mathrm{e}^{\mathrm{i}(\lambda x + \mu y)} F(y,t), \tag{2.4}$$

in which λ and μ are parameters and $F(y,t)$ is a periodic function of y,

$$F(y,t) = \sum_{m=-\infty}^{\infty} a_m(t;\lambda,\mu)\mathrm{e}^{\mathrm{i}my}. \tag{2.5}$$

Substituting (2.3) and (2.4) into (2.2) results in an infinite set of linear differential equations for $a_m(t;\lambda,\mu)$, denoted simply by $a_m(t)$,

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t). \tag{2.6}$$

Here, the infinite-dimensional (column) vector $\boldsymbol{x}(t)$ is, by definition, $\boldsymbol{x}(t) = [\cdots a_{-m}(t) \cdots a_{-1}(t) a_0(t) a_1(t) \cdots a_m(t) \cdots]^{\mathrm{T}}$. The matrix $\boldsymbol{A}$ is defined by

$$\boldsymbol{A} = p_0 \boldsymbol{I} + \boldsymbol{A}_0 + \lambda \boldsymbol{A}_1,$$

where

$$p_0 = -\frac{\lambda^2 + \mu^2}{R},$$

the diagonal matrix $\boldsymbol{A}_0$ is independent of λ,

$$\boldsymbol{A}_0 = -\,\mathrm{diag}\left[\cdots \frac{m^2 - 2m\mu}{R} \cdots \frac{1-2\mu}{R} \; 0 \; \frac{1+2\mu}{R} \cdots \frac{m^2+2m\mu}{R} \cdots\right],$$

and the entries $[\boldsymbol{A}_1]_{ij}$ of perturbation matrix $\boldsymbol{A}_1$ are

$$[\boldsymbol{A}_1]_{ij} = \frac{\lambda^2 + (j+\mu)^2 - (i-j)^2}{\lambda^2 + (i+\mu)^2}(i-j)b_{i-j} \qquad (i,j = 0, \pm 1, \pm 2, \ldots).$$

An important quantity is "the growth rate of instability," σ; by definition,

$$\sigma = \max_{\lambda,\mu}\left\{\max_{\boldsymbol{x}\neq 0} \Re(p) \mid \boldsymbol{A}\boldsymbol{x} = p\boldsymbol{x}\right\}, \tag{2.7}$$

where $\Re(p)$ is the real part of the eigenvalue p.

For a given basic flow, σ is a function of R, and the basic flow is said to be unstable, stable, or neutrally stable if $\sigma > 0$, $\sigma < 0$, or $\sigma = 0$, respectively. We are interested in the critical value R_c of the Reynolds number, such that the instability sets in exactly when R is equal to R_c (and persists for R greater than R_c).

3. Iterative perturbation theory. We are interested here in the growth rate of the "long-wave" Bloch modes; i.e., we want to find asymptotic expressions for the eigenvalue p with maximum real part in the limit $\phi = \sqrt{\lambda^2 + \mu^2} \ll 1$. (For a slightly overcritical Reynolds number, only those modes whose λ and μ are small can be unstable. So, if its λ or μ is not small, the mode will decay.)

We construct a sequence of pairs $(p_i, \boldsymbol{x}_i)$ by the following iteration procedure:

$$\begin{aligned} p_{i+1} &= \lambda \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{x}_i + p_0, \\ \boldsymbol{x}_{i+1} &= (p_{i+1} - p_0) \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{x}_i - \lambda \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{x}_i + \boldsymbol{x}_0 \qquad (i = 0, 1, 2, \ldots). \end{aligned} \tag{3.1}$$

Here $\boldsymbol{x}_0$ is the vector whose 0th element is 1 and all the others are 0: $[\boldsymbol{x}_0]_i = \delta_{0i}$, the Kronecker δ (note that $\boldsymbol{x}_0$ is clearly an eigenvector corresponding to the eigenvalue p_0 for $\lambda = 0$), λ is the perturbation parameter and $\boldsymbol{A}_0^{\mathrm{I}}$ is the generalized inverse of $\boldsymbol{A}_0$. We invoke here the following definition: a matrix $\boldsymbol{B}$ is called the generalized inverse of a given matrix $\boldsymbol{A}$, and denoted $\boldsymbol{A}^{\mathrm{I}}$, if it satisfies the following pair of equations:

$$\begin{aligned} \boldsymbol{ABA} &= \boldsymbol{A}, \\ \boldsymbol{BAB} &= \boldsymbol{B}. \end{aligned} \tag{3.2}$$

It is obvious that if $\boldsymbol{A}$ is invertible, $\boldsymbol{A}^{\mathrm{I}}$ is its inverse in the ordinary sense. In our case,

$$\boldsymbol{A}_0^{\mathrm{I}} = -\operatorname{diag}\left[\cdots \frac{R}{m^2 - 2m\mu} \cdots \frac{R}{1 - 2\mu} \; 0 \; \frac{R}{1 + 2\mu} \cdots \frac{R}{m^2 + 2m\mu} \cdots\right]. \tag{3.3}$$

We prove (see Appendix A) the following theorem.

THEOREM 1. *The iterative sequence* $\{(p_i, \boldsymbol{x}_i)\}$ *generated by* (3.1) *converges to an eigensolution* $(p, \boldsymbol{x})$ *of* $\boldsymbol{Ax} = p\boldsymbol{x}$ *(such that* $\Re(p)$ *is maximum and* $(p, \boldsymbol{x}) \to (p_0, \boldsymbol{x}_0)$ *as* $\lambda \to 0$*), provided* $|\lambda| C \le \sqrt{2}/(4 + 3\sqrt{2})$ *where* C *is a constant,*

$$C = \max\{\|\boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{x}_0\|_2 \; \|\boldsymbol{A}_0^{\mathrm{I}}\|_2, \|\boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1\|_2\}. \tag{3.4}$$

Here, $\|\boldsymbol{x}\|_2$ is the 2-norm of the vector $\boldsymbol{x} = [\cdots \xi_{-n} \cdots \xi_{-1} \xi_0 \xi_1 \cdots \xi_n \cdots]^{\mathrm{T}}$,

$$\|\boldsymbol{x}\|_2 = \left(\sum_n |\xi_n|^2 \right)^{\frac{1}{2}}.$$

Unfortunately, the norm $\|\boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{x}_0\|_2$ in (3.4) is unbounded in any neighborhood of the point $\lambda = \mu = 0$. We can overcome this difficulty as follows. The row vector $(\boldsymbol{A}_1^{\mathrm{T}} \boldsymbol{x}_0)^{\mathrm{T}}$ is the 0th row of $\boldsymbol{A}_1$. We are going to remove the "unbounded part" of this vector by performing a similarity transformation on the full matrix $\boldsymbol{A}$. The transformation matrix

$$\boldsymbol{P} = \boldsymbol{I} + \lambda \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \tag{3.5}$$

can be used, where

$$\boldsymbol{c}^{\mathrm{T}} = \frac{-2\mu}{\lambda^2 + \mu^2} [\cdots m^2 b_m \cdots b_1 \; 0 \; b_{-1} \cdots m^2 b_{-m} \cdots] \tag{3.6}$$

is the unbounded part of the 0th row of $\boldsymbol{A}_1$.

One can see that

$$\boldsymbol{P}^{-1} = \boldsymbol{I} - \lambda \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \tag{3.7}$$

exactly (we have used the fact that $\boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{x}_0 = 0$) and

$$\boldsymbol{P}^{-1} \boldsymbol{A} \boldsymbol{P} = p_0 \boldsymbol{I} + \boldsymbol{A}_0 + \lambda \boldsymbol{P}^{-1} (\boldsymbol{A}_1 - \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}) \boldsymbol{P}. \tag{3.8}$$

We denote

$$\overline{\boldsymbol{A}}_1 = \boldsymbol{P}^{-1} (\boldsymbol{A}_1 - \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}) \boldsymbol{P} \text{ and } \overline{\boldsymbol{A}} = \boldsymbol{P}^{-1} \boldsymbol{A} \boldsymbol{P}, \tag{3.9}$$

so that $\overline{\boldsymbol{A}} = p_0 \boldsymbol{I} + \boldsymbol{A}_0 + \lambda \overline{\boldsymbol{A}}_1$. Hence, Theorem 1 (with $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{A}}_1$ instead of $\boldsymbol{A}$ and $\boldsymbol{A}_1$, respectively) holds. It is not difficult to see that $\|\overline{\boldsymbol{A}}_1^{\mathrm{T}} \boldsymbol{x}_0\|_2$ is bounded. (Indeed, $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ are bounded (due to the factor λ multiplying the unbounded matrix). Also, $\boldsymbol{A}_1 - \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}$ is bounded since the unbounded part of $\boldsymbol{A}_1$ is exactly $\boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}$.) Since $\overline{\boldsymbol{A}} = \boldsymbol{P}^{-1} \boldsymbol{A} \boldsymbol{P}$, the eigenvalues p of $\overline{\boldsymbol{A}} \boldsymbol{x} = p \boldsymbol{x}$ are the same as those of the original problem, $\boldsymbol{A} \boldsymbol{x} = p \boldsymbol{x}$. We use p_2 from (3.1) (with $\overline{\boldsymbol{A}}_1$ in place of $\boldsymbol{A}_1$) to obtain the following approximate expression for the eigenvalue p having the maximum real part:

$$\begin{aligned} p \approx p_2 &= p_0 + \lambda \boldsymbol{x}_0^{\mathrm{T}} \overline{\boldsymbol{A}}_1 \boldsymbol{x}_0 - \lambda^2 \boldsymbol{x}_0^{\mathrm{T}} \overline{\boldsymbol{A}}_1 \boldsymbol{A}_0^{\mathrm{I}} \overline{\boldsymbol{A}}_1 \boldsymbol{x}_0 \\ &= p_0 + \lambda^2 [-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} + (\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})(\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}) + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}], \end{aligned} \tag{3.10}$$

where $\boldsymbol{b} = \boldsymbol{A}_1 \boldsymbol{x}_0$. It follows (see Appendix B) that

$$p_2 \approx p_0 + \lambda^2 (-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}). \tag{3.11}$$

If, alternatively, one uses the iterative formula (3.1) with $\boldsymbol{A}_1$, i.e., without first making the similarity transformation, one arrives at

$$p_2 \approx p_0 + \lambda^2 (-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}). \tag{3.12}$$

Comparing to (3.11), we find that the term $\lambda^3 \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}$ is missing in (3.12). However, the next approximation is

$$\begin{aligned} p_3 &\approx p_2 + \lambda^3 \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} + \lambda^4 (\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})(\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}) \\ &\approx p_2 + \lambda^3 \boldsymbol{x}_0^{\mathrm{T}} (\boldsymbol{A}_1 - \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}} + \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}) \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} \\ &= p_0 + \lambda^2 (-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} + \lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}) + \lambda^3 (\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 - \boldsymbol{c}^{\mathrm{T}}) \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}. \end{aligned} \tag{3.13}$$

(We have neglected the fourth-order term $\lambda^4 (\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})(\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})$ in (3.13).) Thus, we obtain the missing term by splitting the $\boldsymbol{A}_1$ in $\lambda^3 \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}$, which appears only in p_3, into two parts, $\boldsymbol{A}_1 - \boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}$ and $\boldsymbol{x}_0 \boldsymbol{c}^{\mathrm{T}}$. It is easier, especially in time-dependent cases which we will discuss in section 5, to obtain the correct second-order expansion of the perturbed eigenvalue by this "splitting" method than by the similarity-transformation one.

4. Critical parameters of instability. At small λ and μ, we retain only the leading-order terms in (3.11). This yields

$$\Re(p) = -D\phi^2 + O(\phi^4), \tag{4.1}$$

where $\phi^2 = \lambda^2 + \mu^2$, and

$$D(k, R) = \frac{1}{R}\left[1 - \frac{R^2(1-7k^2)}{(1+k^2)^2}B^2 - \frac{2kR^3}{(1+k^2)^2}b^3\right], \tag{4.2}$$

with the following definitions: $k = \mu/\lambda$ (thus, the variables (λ, μ) are changed to variables (k, ϕ)) and

$$B^2 = \sum_{m=-\infty}^{\infty} |b_m|^2, \qquad b^3 = \sum_{m,n} \frac{(m-n)(2n-m)}{mn} b_m^* b_{m-n} b_n. \tag{4.3}$$

(The term with B^2 in (4.2) is the contribution of (B.1) and the one with b^3 is that of (B.4). It is clear that the contributions of (B.2) and (B.3) to (4.1) are $O(\phi^4)$ and therefore do not appear in the expression (4.2) for the "large-eddy viscosity" D.)

For a fixed R, the "viscosity" D, (see (4.2)), depends on the "direction" k. (Clearly, $k = \tan\alpha$, where α is the angle between the quasi wavevector (λ, μ) and the λ-axis.) At some k, $D(k, R)$ has a minimum, and therefore $\frac{\partial D}{\partial k} = 0$. This minimal value of D changes from positive to *negative* as R grows past the critical value R_c; this is the onset of instability. The condition that for $R = R_c$ the viscosity D is zero in the direction k_c for which $D(k, R_c)$ is minimum (and thus nonnegative for all the other directions) yields $D = 0$ and $\frac{\partial D}{\partial k} = 0$, a system (of two equations) which determines both R_c and k_c:

$$(1+k^2)^2 - R^2(1-7k^2)B^2 - 2kR^3b^3 = 0, \tag{4.4}$$

$$2(1+k^2)k + 7R^2kB^2 - R^3b^3 = 0. \tag{4.5}$$

If $b^3 = 0$ (which is clearly the case if (i) the flow has only one harmonic component or (ii) if all b_j are purely imaginary, as is the case for a flow Φ which is odd in an appropriately shifted coordinate system), then $k_c = 0$ from (4.4), and (4.5) yields

$$R_c = \left(\sum_{m=-\infty}^{\infty} |b_m|^2\right)^{-\frac{1}{2}}. \tag{4.6}$$

This expression was also obtained in [20] and [21]. However, those authors did not seem to have recognized the above limitation, $b^3 = 0$, on the applicability of this result. Also, only in the case of $b^3 = 0$, the direction of the critical eddy viscosity is $k = 0$ (which implies $\mu = 0$, i.e., disturbances having the same transverse periodicity as the basic flow). However, the latter was often assumed in the literature to be true in all cases.

The result (4.4) and (4.5) was also obtained in [18] (and was verified by the lattice-gas simulation in [22]). However, as was mentioned in the Introduction, it is not easy to apply the analytic method of [18] to the time-dependent cases. The generalization of our "algebraic" method to such cases is considered in the section which follows.

5. Stability of time-dependent flows. We will restrict our attention to time-dependent basic flows given by "factorizable" stream-functions

(5.1)
$$\Phi(x,y,t) = f(y)g(t) = \sum_{m=-\infty}^{\infty} b_m \mathrm{e}^{\mathrm{i}my} \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}nst}$$
$$(b_0 = 0, b_{-m} = b_m^*, d_{-n} = d_n^*, s \neq 0),$$

where $g(t) = \sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}nst}$.

The corresponding differential equations in vector form are

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t), \tag{5.2}$$

where

$$\boldsymbol{A}(t) = p_0\boldsymbol{I} + \boldsymbol{A}_0 + \lambda\boldsymbol{A}_1(t). \tag{5.3}$$

Here

$$\boldsymbol{A}_1(t) = \left(\sum_{n=-\infty}^{\infty} d_n \mathrm{e}^{\mathrm{i}nst}\right)\boldsymbol{A}_1 \tag{5.4}$$

(see section 2 for the definition of $\boldsymbol{A}_1$).

By analogy with the Floquet theory (see also the discussion in [17]), $\boldsymbol{x}(t)$ should be in the form

$$\boldsymbol{x}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{x}_n \mathrm{e}^{(p+\mathrm{i}ns)t}, \tag{5.5}$$

where p is called the *Floquet exponent*.

Substituting (5.4) and (5.5) into (5.2), we obtain an eigenproblem

$$(p + \mathrm{i}ns)\boldsymbol{x}_n = (p_0\boldsymbol{I} + \boldsymbol{A}_0)\boldsymbol{x}_n + \lambda\boldsymbol{A}_1 \sum_{k=-\infty}^{+\infty} d_{n-k}\boldsymbol{x}_k \qquad (n = 0, \pm1, \pm2, \ldots). \tag{5.6}$$

Equation (5.6) can also be written as

$$\boldsymbol{D}\boldsymbol{y} = p\boldsymbol{y}, \tag{5.7}$$

where $\boldsymbol{y}$ is a block-vector, $\boldsymbol{y} = [\cdots \boldsymbol{x}_{-n}^{\mathrm{T}} \cdots \boldsymbol{x}_{-1}^{\mathrm{T}}\ \boldsymbol{x}_0^{\mathrm{T}}\ \boldsymbol{x}_1^{\mathrm{T}} \cdots \boldsymbol{x}_n^{\mathrm{T}} \cdots]^{\mathrm{T}}$. In (5.7),

$$\boldsymbol{D} = p_0\boldsymbol{I} + \boldsymbol{D}_0 + \lambda\boldsymbol{D}_1,$$

where $\boldsymbol{D}_0$ is a block-diagonal matrix

$$\boldsymbol{D}_0 = \mathrm{diag}[\cdots \boldsymbol{A}_0 + \mathrm{i}ns\boldsymbol{I} \cdots \boldsymbol{A}_0 + \mathrm{i}s\boldsymbol{I}\ \ \boldsymbol{A}_0\ \ \boldsymbol{A}_0 - \mathrm{i}s\boldsymbol{I} \cdots \boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I} \cdots] \tag{5.8}$$

and the block-elements of $\boldsymbol{D}_1$ are

$$[\boldsymbol{D}_1]_{ij} = d_{i-j}\boldsymbol{A}_1 \qquad (i, j = 0, \pm1, \pm2, \ldots). \tag{5.9}$$

Because $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$ have the same structure as $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$, the same method as in section 3 can be used to get

$$p \approx p_0 - \lambda^2\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{D}_1\boldsymbol{D}_0^{\mathrm{I}}\boldsymbol{D}_1\boldsymbol{x}_0 + \lambda^3\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{D}_1\boldsymbol{D}_0^{\mathrm{I}}\boldsymbol{D}_1\boldsymbol{D}_0^{\mathrm{I}}\boldsymbol{D}_1\boldsymbol{x}_0 \tag{5.10}$$

(by replacing $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ with $\boldsymbol{D}_0$ and $\boldsymbol{D}_1$), where $\boldsymbol{D}_0^{\mathrm{I}}$ is the generalized inverse of $\boldsymbol{D}_0$,

$$\boldsymbol{D}_0^{\mathrm{I}} = \mathrm{diag}[\cdots(\boldsymbol{A}_0 + \mathrm{i}ns\boldsymbol{I})^{-1}\cdots(\boldsymbol{A}_0 + \mathrm{i}s\boldsymbol{I})^{-1}\ \boldsymbol{A}_0^{\mathrm{I}}\ (\boldsymbol{A}_0 - \mathrm{i}s\boldsymbol{I})^{-1}\cdots(\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1}\cdots], \tag{5.11}$$

with

$$(\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} = -R\,\mathrm{diag}[\cdots\frac{1}{m^2 - 2m\mu - \mathrm{i}nsR}\cdots\frac{1}{1 - 2\mu - \mathrm{i}nsR}\cdots \frac{1}{\mathrm{i}nsR}\cdots\frac{1}{1 + 2\mu + \mathrm{i}nsR}\cdots\frac{1}{m^2 + 2m\mu + \mathrm{i}nsR}\cdots]. \tag{5.12}$$

The real part of p, $\Re(p)$, can be expressed as follows (see Appendix C):

$$\Re(p) = -D\phi^2 + O(\phi^4), \tag{5.13}$$

where

$$D = \frac{1}{R}\left\{1 - \frac{R^2 f_1}{(1+k^2)^2} + \frac{8k^2R^2 f_2}{(1+k^2)^2} - \frac{2kR^3 f_3}{(1+k^2)^2}\right\}, \tag{5.14}$$

$$f_1 = \sum_{m,n}\frac{m^4|b_m|^2|d_n|^2}{m^4 + n^2s^2R^2}, \tag{5.15}$$

$$f_2 = \sum_{m,n}\frac{m^8|b_m|^2|d_n|^2}{(m^4 + n^2s^2R^2)^2}, \tag{5.16}$$

$$f_3 = \sum_{j,l,m,n}(b_{-j}b_{j-l}b_l)(d_{-m}d_{m-n}d_n)\frac{jl(2l-j)(j-l)(j^2l^2 - mns^2R^2)}{(j^4 + m^2s^2R^2)(l^4 + n^2s^2R^2)}. \tag{5.17}$$

The critical values of parameters for the onset of instability, Reynolds number R_c and direction k_c, can be obtained from the equations $D = 0$ and $\frac{\partial D}{\partial k} = 0$; that is,

$$(1+k^2)^2 - (1+k^2)R^2 f_1 + 8k^2R^2 f_2 - 2kR^3 f_3 = 0, \tag{5.18}$$

$$2k(1+k^2) - kR^2 f_1 + 8kR^2 f_2 - R^3 f_3 = 0. \tag{5.19}$$

Equations (5.18) and (5.19) determine the two implicit functions, $R_c(s)$ and $k_c(s)$. We want to know how the stability properties change with s. We first consider two different limiting cases, $s \to 0$ and $s \to \infty$.

In the case of $s \to 0$, one obtains (assuming that the basic flow has an arbitrary but *finite* number of nonzero Fourier coefficients d_n)

$$\Re(p) = -\frac{\phi^2}{(1+k^2)^2R}\left\{(1+k^2)^2 - R^2(1-7k^2)B^2\sum_n|d_n|^2 - 2kR^3b^3\sum_{m,n}(d_{-m}d_{m-n}d_n)\right\} \qquad (s \to 0), \tag{5.20}$$

where the constant B^2 and b^3 have been defined by (4.3). The equations which determine the critical parameters, (5.18) and (5.19), take the simpler, polynomial (in R and k) form

$$(1+k^2)^2 - R^2(1-7k^2)B^2\sum_n |d_n|^2 - 2kR^3b^3\sum_{m,n}(d_{-m}d_{m-n}d_n) = 0, \tag{5.21}$$

$$2k(1+k^2) + 7kR^2B^2\sum_n |d_n|^2 - 2R^3b^3\sum_{m,n}(d_{-m}d_{m-n}d_n) = 0. \tag{5.22}$$

These equations are simple to solve numerically, obtaining the critical values of R and k. It must be pointed out that these critical values yield the limit of R_c and k_c as $s \to 0$ for STP basic flows. They are different from the values of the critical parameters at $s = 0$ (discussed in section 3): when $s = 0$, the basic flow is no longer time-dependent and therefore the Floquet–Bloch modes (5.5) are not relevant any more.

In the case $s \to \infty$ we have

(5.23)

$$\Re(p) = \frac{-\phi^2}{(1+k^2)^2R}[(1+k^2)^2 - R^2B^2d_0^2 + 7k^2R^2B^2d_0^2 - 2kR^3d_0^3b^3] + O(\phi^4)\ (s\to\infty).$$

Comparing this with (4.1) we can see that (5.23) is, in fact, the second-order approximation of the growth rate of a disturbance of the time-independent basic flow given by the stream-function

$$\Phi(x,y,t) = d_0\sum_{m=-\infty}^{\infty} b_m \mathrm{e}^{imy} \qquad (b_0 = 0, b_{-m} = b_m^*). \tag{5.24}$$

It follows that, in this case, the stability is determined by the *time-averaged* basic flow. The corresponding equations which determine the critical parameters R_c and k_c (when $d_0 \neq 0$) are

$$(1+k^2)^2 - R^2B^2d_0^2 + 7k^2R^2B^2d_0^2 - 2kR^3d_0^3b^3 = 0, \tag{5.25}$$

$$2k(1+k^2) + 7kR^2B^2d_0^2 - R^3d_0^3b^3 = 0. \tag{5.26}$$

It is easy to see that if R_c and k_c are the solutions of (4.4) and (4.5) then $R_c/|d_0|$ and $k_cd_0/|d_0|$ are the solutions of (5.25) and (5.26). When $d_0 = 0$, that is, the time-averaged flow is zero, then

$$\Re(p) = -\frac{\phi^2}{R} + O(\phi^4) < 0 \qquad (s\to\infty). \tag{5.27}$$

Thus, such flows are stable for any R.

We denote the critical values of the Reynolds number corresponding to the limits $s = 0$ and $s = \infty$ as $R_c^{(0)}$ and $R_c^{(\infty)}$, respectively; similarly, the critical directions are $k_c^{(0)}$ and $k_c^{(\infty)}$. Clearly, R_c changes from $R_c^{(0)}$ to $R_c^{(\infty)}$ and k_c from $k_c^{(0)}$ to $k_c^{(\infty)}$ as s changes from 0 to ∞, provided $d_0 \neq 0$. However, when $d_0 = 0$, there exists a value s_c such that $R_c \to \infty$ as $s \to s_c$ (see (5.23)). The value s_c is thus the critical frequency: if $s > s_c$ the flow is stable for any Reynolds number R; if $s < s_c$ there exists a critical

Reynolds number R_c, such that the flow is unstable iff $R > R_c$. (This generalizes the previous observation [16] of such absolute stabilization for some special simple flows.)

We want to derive the asymptotic expressions for $R_c(s)$ and $k_c(s)$ (the subscript c will be omitted in the following discussion for brevity) (i) as $s \downarrow 0$ (i.e., s goes to 0 from above) and (ii) as $s \uparrow s_c$.

When $s \downarrow 0$, we write R^2 and k^2 in the form of perturbation series:

$$R^2 = R_0^2 + R_1 s^2 + O(s^4), \tag{5.28}$$

$$k^2 = k_0^2 + k_1 s^2 + O(s^4). \tag{5.29}$$

One can see that R_0 and k_0 are actually the solution of (5.21) and (5.22), i.e., $R_0 = R_c^{(0)}$ and $k_0 = k_c^{(0)}$. The correction coefficients R_1 and k_1 are found in Appendix D.

For the other limit, $s \uparrow s_c$, it can be shown (see Appendix E) that the approximate expressions of R_c and k_c are

$$R = \frac{s_1}{\sqrt{s_c^2 - s^2}} \qquad (s \uparrow s_c), \tag{5.30}$$

$$k = \frac{k_1\sqrt{s_c^2 - s^2}}{s_1} \qquad (s \uparrow s_c). \tag{5.31}$$

The constants, s_c, s_1, and k_1 are expressed in terms of the basic flow (see Appendix E) as

$$s_c^2 = \sum_m m^4 |b_m|^2 \sum_{n\neq 0} \frac{|d_n|^2}{n^2}, \tag{5.32}$$

$$k_1 = \frac{1}{s_c^4} f_3^{(2)}, \tag{5.33}$$

$$s_1 = \frac{1}{s_c}\left[f_1^{(2)} - \frac{1}{s_c^4}(f_3^{(2)})^2\right]^{\frac{1}{2}}, \tag{5.34}$$

$$f_1^{(2)} = f_2^{(1)} = \sum_m m^8 |b_m|^2 \sum_{n\neq 0} \frac{|d_n|^2}{n^4}, \tag{5.35}$$

$$f_3^{(2)} = \sum_{\substack{j,l,m,n\\(m,n\neq 0)}} \frac{(b_{-j}b_{j-l}b_l)(d_{-m}d_{m-n}d_n)jl(2l-j)(j-l)(mnj^2l^2 + n^2j^4 + m^2l^4)}{m^3n^3}. \tag{5.36}$$

Thus, we have the two-term asymptotic expressions for $R_c(s)$ and $k_c(s)$ near $s = 0$ (see (5.28) and (5.29)) and the leading-order asymptotic expressions near $s = s_c$ (for $d_0 = 0$; see (5.30) and (5.31)).

In general, it is impossible to solve (5.18) and (5.19) for the critical parameters *analytically*. However, for certain subclasses of flows some analytical treatment is possible. As an example, we consider the case

$$f_3 = 0. \tag{5.37}$$

We will say that a function $f(t)$ is an essentially even (odd) function if there exists t_0 such that $f(t - t_0)$ is an even (odd) function. Equation (5.37) is true if

(i) one of the functions $f(y)$ and $g(t)$ is an essentially even and the other an essentially odd function (in this case every product $(b_{-j}b_{j-l}b_l)(d_{-m}d_{m-n}d_n)$ is a purely imaginary number), or

(ii) $b^*_{-m_o} = b_{m_o} \neq 0$ ($m_o \neq 0$) and $b_m = 0$ ($m \neq m_o$), or $d^*_{-n_o} = d_{n_o} \neq 0$ ($n_0 \neq 0$) and $d_n = 0$ ($n \neq n_0$), which means that at least one of the factors $f(y)$ and $g(t)$ of the basic flow (5.1) has exactly one harmonic term (so that $(b_{-j}b_{j-l}b_l)(d_{-m}d_{m-n}d_n) = 0$ in (5.36)).

The equations to determine the critical parameters are now

$$(1+k^2)^2 - R^2 f_1 - k^2 R^2 f_1 + 8k^2 R^2 f_2 = 0, \tag{5.38}$$

$$2k(1+k^2) - kR^2 f_1 + 8kR^2 f_2 = 0. \tag{5.39}$$

It is obvious that $k = 0$ and R_c, which is determined by

$$R_c^2 \sum_{m,n} \frac{m^4 |b_m|^2 |d_n|^2}{m^4 + n^2 s^2 R_c^2} = 1, \tag{5.40}$$

is a solution, and furthermore, we are going to prove that this solution of (5.38) and (5.39) is unique. Indeed, any other possible solution would have $k \neq 0$ and hence must satisfy the system

$$(1+k^2)^2 - R^2 f_1 - k^2 R^2 f_1 + 8k^2 R^2 f_2 = 0, \tag{5.41}$$

$$2(1+k^2) - R^2 f_1 + 8R^2 f_2 = 0. \tag{5.42}$$

The subtraction of (5.42) multiplied by k^2 from (5.41) gives

$$R^2 f_1 = (1+k^2)(1-k^2). \tag{5.43}$$

From (5.42) and (5.43) we have

$$(1+k^2)^2 + 8R^2 f_2 = 0. \tag{5.44}$$

However, the left-hand side is strictly positive for any R and k because f_2 is always positive. So we have proved that (5.38) and (5.39) have the unique solutions $k = 0$ and R_c satisfying (5.40); therefore, the solutions with $k \neq 0$ appear only when $f_3 \neq 0$. Now, we have

$$\frac{dR}{ds} = -\frac{\dfrac{\partial}{\partial s}(R^2 f_1)}{\dfrac{\partial}{\partial R}(R^2 f_1)} = \frac{sR^3 C}{f_2} > 0 \qquad (s > 0), \tag{5.45}$$

where

$$C = \sum_{\substack{m,n \\ (n\neq 0)}} \frac{m^4 n^2 |b_m|^2 |d_n|^2}{(m^4 + n^2 s^2 R^2)^2}. \tag{5.46}$$

This means that $R(s)$ is a monotonously increasing function.

The approximate expressions of $R(s)$ are now

$$R = \frac{1}{\sqrt{a_0}} \sqrt{1 + \frac{a_1 s^2}{a_0^2}} \qquad (s \downarrow 0) \tag{5.47}$$

and

$$R = \frac{\sqrt{f_1^{(2)}}}{\sqrt{1 - \dfrac{s^2}{s_c^2}}} \qquad (s \uparrow s_c). \tag{5.48}$$

6. Some examples. In this section, we consider several specific flows whose stability properties follow from the general theory of the preceding section.

Example 1.

$$\Phi(x, y, t) = \cos(y - \theta) + \cos(2y - \theta) \qquad (0 \le \theta \le \pi), \tag{6.1}$$

a time-independent flow. Clearly, $b_1 = b_{-1}^* = b_2 = b_{-2}^* = \frac{1}{2}\,\mathrm{e}^{-i\theta}$ and $B^2 = 1$ and $b^3 = \frac{9}{8}\cos\theta$. This is perhaps the simplest example of a flow exhibiting the *nontransverse* large-scale instability, that is, $k_c \neq 0$. Figure 1 shows that R_c and k_c (obtained by solving numerically the system of (4.4) and (4.5)) vary with the phase shift θ.

Example 2.

$$\Phi(x, y, t) = \cos(y)[d_0 + \cos(t)]. \tag{6.2}$$

This is an example demonstrating the influence of d_0 on the stability of the basic flow. It is easy to see that $f_3 = 0$ for this flow. Also, $k_c = 0$, and therefore R_c as a function of s is determined, from (5.40), by the following equation:

$$\frac{R_c^2}{2} d_0^2 + \frac{1}{4}\left(\frac{R_c^2}{1 + s^2 R_c^2}\right) = 1. \tag{6.3}$$

When $d_0 = 0$, we obtain the simple expression

$$R_c = \left(\frac{1}{4} - s^2\right)^{-\frac{1}{2}} \qquad \left(0 \le s < \frac{1}{2}\right). \tag{6.4}$$

This result was first obtained in [16].

It is clear from (6.4) that the critical frequency is

$$s_c = \frac{1}{2}. \tag{6.5}$$

For $d_0 \neq 0$, we have

$$R_c^2 = \frac{\sqrt{\left(\frac{d_0^2}{2} + \frac{1}{4} - s^2\right)^2 + 2d_0^2 s^2} - \left(\frac{d_0^2}{2} + \frac{1}{4} - s^2\right)}{d_0^2 s^2}. \tag{6.6}$$

The asymptotic expressions near the singular points, $s = 0$ and $s = \infty$, are

$$R_c^2 = \frac{4}{2d_0^2 + 1}\left(1 + \frac{4s^2}{(2d_0^2 + 1)^2}\right) + O(s^4) \qquad (s \downarrow 0) \tag{6.7}$$

and

$$R_c^2 = \frac{2}{d_0^2} \qquad (s \to \infty). \tag{6.8}$$

From (6.6), we also have

$$R_c^2\big|_{s = s_c} = 2\left(\frac{\sqrt{2}}{|d_0|} - 1\right) \qquad (d_0 \downarrow 0). \tag{6.9}$$

It can be seen from (6.9) that there is a big change at $s = s_c$ when $d_0 \downarrow 0$. Figure 2 shows the curves of $d_0 R_c$ for different d_0. (In Fig. 2, $d_0 R_c$ is pictured instead of R_c in order to fit all the curves in one figure.) When $d_0 \downarrow 0$, we have an asymptotic representation for $d_0 R_c$:

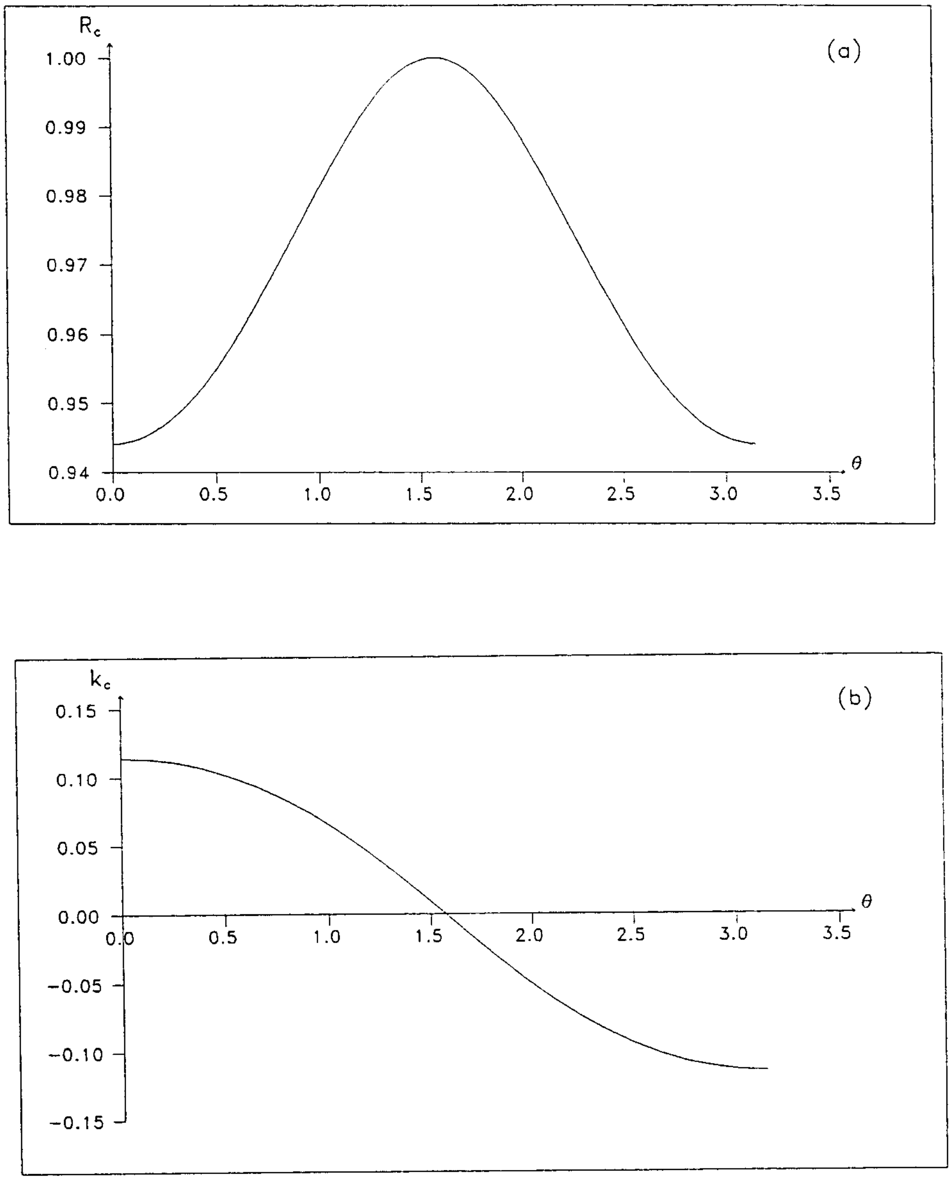

FIG. 1. *Dependence of the critical parameters on the base angle* θ *of the basic flow* $\Phi(x, y, t) = \cos(y - \theta) + \cos(2y - \theta)$ $(0 \leq \theta \leq \pi)$: (a) *the critical Reynolds number* R_c *and* (b) *the critical wave-direction number* k_c.

$$d_0 R_c \to \begin{cases} 0, & 0 \leq s \leq \dfrac{1}{2} \\ \left[2\left(1 - \dfrac{1}{4s^2}\right)\right]^{\frac{1}{2}}, & s > \dfrac{1}{2} \end{cases} \quad (d_0 \downarrow 0). \tag{6.10}$$

Example 3.

$$\Phi(x, y, t) = [\cos(y) + \cos(2y)][d_0 + \cos(t) + \cos(2t)]. \tag{6.11}$$

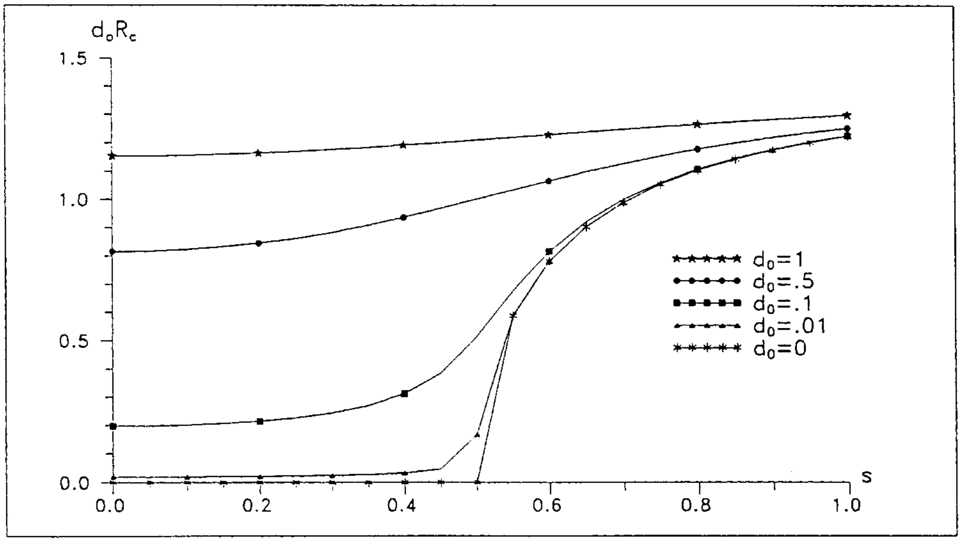

FIG. 2. *Scaled critical Reynolds number* ($d_0 R_c$) *as a function of the frequency* s *for the basic flow* $\Phi(x, y, t) = \cos y(d_0 + \cos st)$.

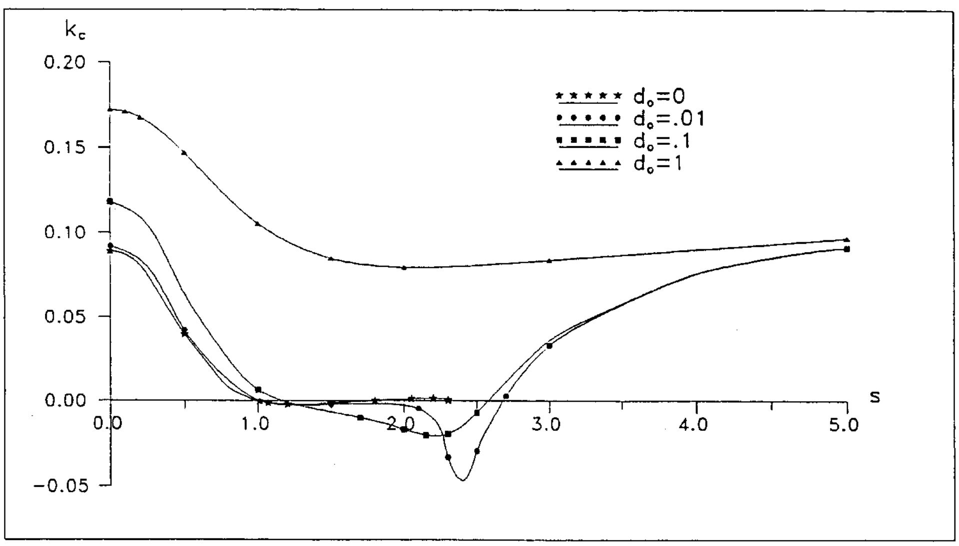

FIG. 3. *Critical wave-direction number* k_c *as a function of the frequency* s *for the basic flow* $\Phi(x, y, t) = (\cos y + \cos 2y)(d_0 + \cos st + \cos 2st)$.

This is a simple example of the nontransverse large-scale instability for STP flow. From (5.17), $f_3 \neq 0$. The results of stability analysis, solving numerically the system (5.18)–(5.19), are shown in Figs. 3 and 4.

In Fig. 4, we see that R_c is an increasing function of s. It is tempting to speculate that this is a general property which holds far beyond the particular case $f_3 = 0$ (see (5.45)); however, we have not been able to prove that this is the case.

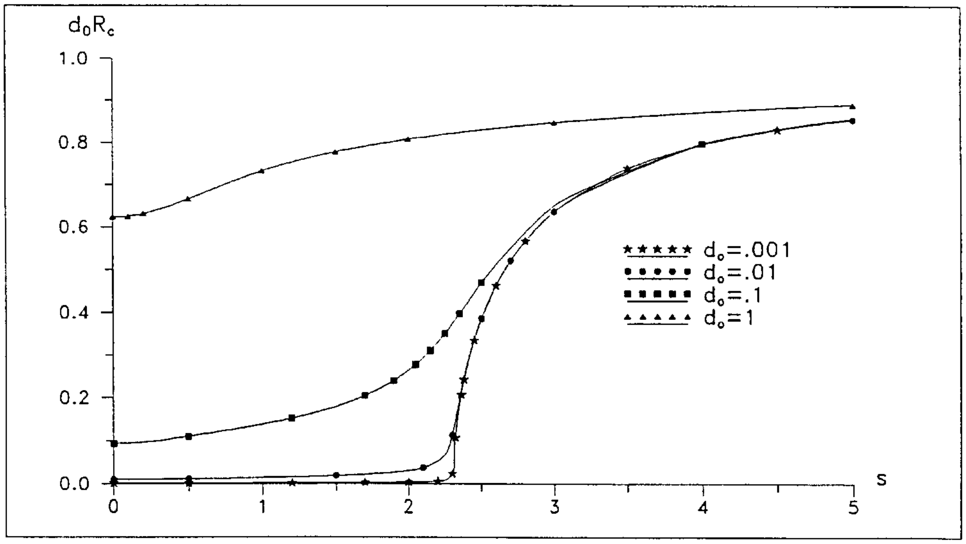

FIG. 4. *Scaled critical Reynolds number* (d_0R_c) *as a function of the frequency s for the basic flow* $(\cos y + \cos 2y)(d_0 + \cos st + \cos 2st)$.

7. Summary and concluding remarks. We have studied the stability of two-dimensional incompressible flows periodic in space and either steady or periodic in time. Our research shows that by introducing the usual (Bloch) normal modes in the steady case and the quasi-normal, "Floquet–Bloch" modes for the time-periodic case, the question of linear stability reduces from a PDE to a more convenient, *algebraic* eigenvalue problem. Using this universal approach, we have obtained unified results for large classes of flows. A number of seemingly disconnected results previously documented in the literature turn out to be just particular cases of our more general formulation.

For the unidirectional flows we considered, the infinite-dimensional eigenvalue-problem matrix was found to have a special structure. We used this to construct an iteration method which we rigorously proved to *converge* and yield the first (i.e., maximum real part) eigenvalue for sufficiently small Bloch wavenumbers. In this way, we established that the large-scale instability has a diffusive character and found the general expression of the eddy-diffusion coefficient in terms of the basic-flow Fourier-coefficients. From this, the problem of determining the critical parameters, the Reynolds number R_c and the wavevector direction k_c, has been reduced to a system of two algebraic equations. We find that $k_c \neq 0$ in general; i.e., the large-scale instability is nontransversal, which corroborates the result obtained recently by others who used a more complicated, differential-operator method. We find that our approach is more versatile: in contrast to the other one, it allows one to find numerically the growth rate for any wavenumbers, rather than only small ones. (This is especially important for multidirectional flows, where the critical wavelength of instability turns out to be comparable to the basic-flow lengthscale.)

We obtained and analyzed the eddy-diffusivity expression for a general class of STP unidirectional flows, whereas previously only a few simple particular examples of time-dependent periodic flows were treated in the literature. The most interesting result is that any flow of this class becomes absolutely stable for sufficiently high frequency of oscillation, $s > s_c$. We have obtained a general expression for the critical frequency s_c. When $s \uparrow s_c$, we have $k_c(s) \to 0$, which means that only the transverse instability takes place. We have obtained the asymptotic expressions of $R_c(s)$ and $k_c(s)$ at $s \downarrow 0$ and $s \uparrow s_c$.

For a certain class of basic flows (defined by the condition $f_3 = 0$), we have shown that $k_c(s) \equiv 0$ and $R_c(s)$ is a monotonously increasing function. It is tempting to speculate that the latter property might be inherent in *any* periodic flow (rather than being restricted to only some special classes of such flows). Also, one can think of applying the algebraic fundamental-mode approach of this paper to *three-dimensional* periodic flows, unidirectional flows of *non-Newtonian* fluids, etc. More detail can be found in [25].

Appendix A. First, we can prove the following two lemmas.

LEMMA 1. *Any solution of a* consistent *linear system (i.e., a system which has at least one solution)*

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{A.1}$$

can be expressed as

$$\boldsymbol{x} = \boldsymbol{A}^{\mathrm{I}}\boldsymbol{b} + \boldsymbol{x}_g, \tag{A.2}$$

where $\boldsymbol{x}_g$ is some solution of the homogeneous equations; i.e., $\boldsymbol{x}_g \in \mathrm{Null}(\boldsymbol{A})$, the null space of $\boldsymbol{A}$, which is defined as

$$\mathrm{Null}(\boldsymbol{A}) = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}. \tag{A.3}$$

(In (A.2), $\boldsymbol{A}^{\mathrm{I}}$ is *any* of the generalized inverses of $\boldsymbol{A}$, e.g., $\boldsymbol{A}^{\mathrm{I}}$ defined by (3.2)).

The proof is simple and omitted here.

LEMMA 2. *If we normalize $\boldsymbol{x}$ so that the center element of $\boldsymbol{x}$ is 1, then the equation*

$$p_0\boldsymbol{x} + \boldsymbol{A}_0\boldsymbol{x} + \lambda\boldsymbol{A}_1\boldsymbol{x} = p\boldsymbol{x} \tag{A.4}$$

and the system

$$\begin{aligned} p &= \lambda\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x} + p_0, \\ \boldsymbol{x} &= \boldsymbol{A}_0^{\mathrm{I}}[(p-p_0)\boldsymbol{x} - \lambda\boldsymbol{A}_1\boldsymbol{x}] + \boldsymbol{x}_0 \end{aligned} \tag{A.5}$$

are equivalent (*where p_0, $\boldsymbol{A}_0$, and $\boldsymbol{A}_1$ are given in section* 3 *and $\boldsymbol{x}_0$ is a vector whose center element is* 1 *and all others are* 0).

Proof. It is easy to see that $\boldsymbol{x}_0$ is a basis of the null space of $\boldsymbol{A}_0$ (and of $\boldsymbol{A}_0^{\mathrm{I}}$ as well, which will be used below). Suppose p and $\boldsymbol{x}$ are such that (A.4) is satisfied. By Lemma 1, with $\boldsymbol{A} = \boldsymbol{A}_0, \boldsymbol{b} = [(p-p_0)\boldsymbol{x} - \lambda\boldsymbol{A}_1\boldsymbol{x}]$, and $\boldsymbol{x}_g = c_1\boldsymbol{x}_0$, we have

$$\boldsymbol{x} = \boldsymbol{A}_0^{\mathrm{I}}[(p-p_0)\boldsymbol{x} - \lambda\boldsymbol{A}_1\boldsymbol{x}] + c_1\boldsymbol{x}_0. \tag{A.6}$$

The central element of $\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{y}$ is 0 for any $\boldsymbol{y}$ and, by the normalization condition, we have $c_1 = 1$. Multiplication by $\boldsymbol{x}_0^{\mathrm{T}}$ to both sides of (A.4) gives

$$p = \lambda\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x} + p_0, \tag{A.7}$$

because from the definition of $\boldsymbol{x}_0$ and $\boldsymbol{A}_0$, we have $[\boldsymbol{x}_0]_m = \delta_{0m}$, and also $\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_0 = \boldsymbol{0}$ and $\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{x} = 1$. Then (A.6) and (A.7) show that the solutions p and $\boldsymbol{x}$ of (A.4) satisfy (A.5).

Conversely, if p and $\boldsymbol{x}$ are the solutions of (A.5), then from the second equation of (A.5), by applying Lemma 1 with $\boldsymbol{A}_0^{\mathrm{I}}$ playing the roll of $\boldsymbol{A}$ in (A.1), $\boldsymbol{x}-\boldsymbol{x}_0$ the role of $\boldsymbol{b}$, and $c_2\boldsymbol{x}_0$ the role of $\boldsymbol{x}_g$ in (A.2), we have

$$(p-p_0)\boldsymbol{x}-\lambda\boldsymbol{A}_1\boldsymbol{x}=\boldsymbol{A}_0\boldsymbol{x}+c_2\boldsymbol{x}_0. \tag{A.8}$$

(We have used the readily seen facts that, for any $\boldsymbol{B}$, $(\boldsymbol{B}^{\mathrm{I}})^{\mathrm{I}}=\boldsymbol{B}$, and also, $\boldsymbol{A}_0\boldsymbol{x}_0=0$.) Multiplying both sides of (A.8) by $\boldsymbol{x}_0^{\mathrm{T}}$ and using the first equation of (A.5), we obtain $c_2=0$. Hence, (A.8) becomes (A.4). Lemma 2 has now been proved. □

Substituting the first equation into the second one, we write (A.5) in the equivalent form

$$\begin{aligned} p&=\lambda\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x}+p_0,\\ \boldsymbol{x}&=\lambda\boldsymbol{A}_0^{\mathrm{I}}[(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x})\boldsymbol{x}-\boldsymbol{A}_1\boldsymbol{x}]+\boldsymbol{x}_0. \end{aligned} \tag{A.9}$$

Thus, if solution $(p,\boldsymbol{x})$ of (A.9) exists and $\boldsymbol{x}\in l^r$, then $\boldsymbol{x}$ is the *fixed point* of *mapping* $\boldsymbol{F}(\boldsymbol{x})$ on l^r, where $\boldsymbol{F}(\boldsymbol{x})$ is given by

$$\boldsymbol{F}(\boldsymbol{x})=\lambda\boldsymbol{A}_0^{\mathrm{I}}[(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x})\boldsymbol{x}-\boldsymbol{A}_1\boldsymbol{x}]+\boldsymbol{x}_0. \tag{A.10}$$

As is well known, l^r is a vector space whose element $\boldsymbol{x}=[\cdots\xi_{-n}\cdots\xi_{-1}\xi_0\xi_1\cdots\xi_n\cdots]^{\mathrm{T}}$ satisfies

$$\sum_n|\xi_n|^r<\infty\qquad(1<r<\infty). \tag{A.11}$$

The norm on l^r is defined as

$$\|\boldsymbol{x}\|_r=\left(\sum_n|\xi_n|^r\right)^{\frac{1}{r}}. \tag{A.12}$$

We will use the well-known definition and theorem of contractive mapping (e.g., [23, p. 125] and [24, p. 66]). The definition and the theorem are originally formulated for any metric space. For the purposes of our discussion, we recast those in terms of the particular case, l^r.

Definition. *A mapping $\boldsymbol{f}(\boldsymbol{x})$ on l^r is said to be a contractive mapping, if there is a real number $0\le L<1$ such that for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in l^r*

$$\|\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{y})\|_r\le L\|\boldsymbol{x}-\boldsymbol{y}\|_r. \tag{A.13}$$

Theorem 2 (contractive mapping theorem). *Let X be a complete subspace of l^r and let $\boldsymbol{f}\colon X\to X$ be a contractive mapping. Then there is one and only one point $\boldsymbol{x}^*$ in X such that*

$$\boldsymbol{f}(\boldsymbol{x}^*)=\boldsymbol{x}^*. \tag{A.14}$$

Moreover, if $\boldsymbol{x}_1$ is any point in X and $\boldsymbol{x}_n$ is defined inductively by $\boldsymbol{x}_{n+1}=\boldsymbol{f}(\boldsymbol{x}_n)$ $(n=1,2,\ldots)$, then $\boldsymbol{x}_n\to\boldsymbol{x}^*$ as $n\to\infty$. (That is, $\boldsymbol{f}$ has a unique fixed point $\boldsymbol{x}^*$, and every sequence of iterations of $\boldsymbol{f}$ converges to this fixed point.)

Now we prove Theorem 1 of section 3. We first prove that if $|\lambda|C<\sqrt{2}/(4+3\sqrt{2})$, then $\boldsymbol{F}$ given by (A.10) is a contractive mapping on the complete subspace X_M of l^r, where $M=1+\sqrt{2}$,

$$X_M=\{\boldsymbol{x}\mid\|\boldsymbol{x}\|_r\le M\}, \tag{A.15}$$

and C has the form which is more general than that in the theorem,

$$C = \max\{\|\boldsymbol{A}_0^{\mathrm{I}}\|_r \,\|\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{x}_0\|_q, \|\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1\|_r\} \qquad \left(\frac{1}{r}+\frac{1}{q}=1\right). \tag{A.16}$$

Proof. (1) For any $\boldsymbol{x} \in X_M$,

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{x})\|_r &= \|\lambda \boldsymbol{A}_0^{\mathrm{I}}[(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x})\boldsymbol{x} - \boldsymbol{A}_1\boldsymbol{x}] + \boldsymbol{x}_0\|_r \\ &\le |\lambda|(\|\boldsymbol{A}_0^{\mathrm{I}}\|_r \,\|\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{x}_0\|_q\|\boldsymbol{x}\|_r\|\boldsymbol{x}\|_r + \|\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1\|_r\|\boldsymbol{x}\|_r) + \|\boldsymbol{x}_0\|_r \\ &\le |\lambda| C(M^2+M) + 1 < \frac{\sqrt{2}}{4+3\sqrt{2}}[(1+\sqrt{2})^2 + 1 + \sqrt{2}] + 1 = 1+\sqrt{2} = M,\end{aligned} \tag{A.17}$$

that is, $\boldsymbol{F}(\boldsymbol{x}) \in X_M$. In (A.17), Hölder's inequality (e.g., [24, p. 41])

$$|\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}| \le \|\boldsymbol{x}\|_q\|\boldsymbol{y}\|_r \qquad \left(\text{for any } r \text{ and } q \text{ such that } \frac{1}{r}+\frac{1}{q}=1\right) \tag{A.18}$$

has been used

(2) For any $\boldsymbol{x}, \boldsymbol{y} \in X_M$, we have

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})\|_r &= |\lambda| \,\|\boldsymbol{A}_0^{\mathrm{I}}[(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x})\boldsymbol{x} - (\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{y})\boldsymbol{y}] - \boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1(\boldsymbol{x}-\boldsymbol{y})\|_r \\ &\le |\lambda| \,\{\|\boldsymbol{A}_0^{\mathrm{I}}(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x})\boldsymbol{x} - (\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{y})\boldsymbol{x}]\|_r \\ &\qquad + \|\boldsymbol{A}_0^{\mathrm{I}}[(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{y})\boldsymbol{x} - (\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{y})\boldsymbol{y}]\|_r + \|\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1(\boldsymbol{x}-\boldsymbol{y})\|_r\} \\ &\le |\lambda|[[\|\boldsymbol{A}_0^{\mathrm{I}}\|_r \,\|\boldsymbol{A}_1^{\mathrm{T}}\boldsymbol{x}_0\|_q(\|\boldsymbol{x}\|_r + \|\boldsymbol{y}\|_r) + \|\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1\|_r]\,\|\boldsymbol{x}-\boldsymbol{y}\|_r \\ &\le |\lambda| C(2M+1)\|\boldsymbol{x}-\boldsymbol{y}\|_r = L\|\boldsymbol{x}-\boldsymbol{y}\|_r,\end{aligned} \tag{A.19}$$

where

$$L = |\lambda| C(2M+1) < \frac{\sqrt{2}}{4+3\sqrt{2}}\,[2(1+\sqrt{2})+1] = 1. \tag{A.20}$$

We see that $\boldsymbol{F}$ is a contractive mapping on X_M; therefore, by the contractive mapping theorem, for any $\boldsymbol{x}_1 \in X_M$ the sequence

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}(\boldsymbol{x}_i) = \lambda(\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x}_i)\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{x}_i - \lambda\boldsymbol{A}_0^{\mathrm{I}}\boldsymbol{A}_1\boldsymbol{x}_i + \boldsymbol{x}_0 \qquad (i=1,2,\ldots) \tag{A.21}$$

converges to the unique fixed point $\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x})$.

(The bound $|\lambda| C < \sqrt{2}/(4+3\sqrt{2})$ and $M = 1+\sqrt{2}$ can be obtained as follows. In (A.17), we want

$$|\lambda| C(M^2+M) + 1 < M, \tag{A.22}$$

that is,

$$|\lambda| C < \frac{M-1}{M^2+M}. \tag{A.23}$$

It is easy to see that $\frac{M-1}{M^2+M}$ reaches its maximum $\sqrt{2}/(4+3\sqrt{2})$ at $M = 1+\sqrt{2}$.)

To construct a sequence converging to the quantity p appearing in the first equation of (A.9), we set

$$p_{i+1} = \lambda\boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{A}_1\boldsymbol{x}_i + p_0 \qquad (i=1,2,\ldots). \tag{A.24}$$

Since $\boldsymbol{x}_i$ of (A.21) converges to $\boldsymbol{x}$, it is clear that

$$p_i \to p \qquad (i \to \infty) \tag{A.25}$$

and (A.9) is satisfied by $p = \lim_{i\to\infty}$ and $\boldsymbol{x} = \lim_{i\to\infty} \boldsymbol{x}_i$.

Since $\|\boldsymbol{x}_0\|_r = 1$, it is clear that $\boldsymbol{x}_0 \in X_M$, and we can start the iteration from $\boldsymbol{x}_0$.

By letting $r = 2$ (therefore $q = 2$), we obtain Theorem 1.

Appendix B. The terms of (3.10) can be written as the following series:

(B.1)

$$-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} = \frac{2R}{\lambda^2+\mu^2} \sum_{m=1}^{\infty} m^2 |b_m|^2 \frac{(m^2-\lambda^2-\mu^2)[(\lambda^2+\mu^2)^2+m^2(\lambda^2-7\mu^2)]}{(m^2-4\mu^2)[(\lambda^2+\mu^2+m^2)^2-4m^2\mu^2]},$$

(B.2)

$$\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} = \frac{8\lambda\mu^2 R}{\lambda^2+\mu^2} \sum_{m=1}^{\infty} m^2 |b_m|^2 \frac{(m^2-\lambda^2-\mu^2)(\lambda^2+\mu^2+2m^2)}{(m^2-4\mu^2)[(\lambda^2+\mu^2+m^2)^2-4m^2\mu^2]},$$

(B.3)

$$\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} = \frac{-8\lambda\mu^2 R^2}{\lambda^2+\mu^2} \sum_{m=1}^{\infty} m^2 |b_m|^2 \frac{(m^2-\lambda^2-\mu^2)(2\lambda^2+6\mu^2+3m^2)}{(m^2-4\mu^2)^2[(\lambda^2+\mu^2+m^2)^2-4m^2\mu^2]},$$

(B.4)

$$\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b} = \frac{2\lambda\mu R^2}{\lambda^2+\mu^2} \sum_{m,n=-\infty}^{\infty} \frac{m^2 n[\lambda^2+(n+\mu)^2-(m-n)^2](m-n)(n^2-\lambda-\mu^2)}{(m^2+2m\mu)(n^2+2n\mu)[\lambda^2+(m+\mu)^2][\lambda^2+(n+\mu)^2]} b_m^* b_{m-n} b_n.$$

From these expansions, it is obvious that $-\boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}$ and $\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_1 \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b}$ in (3.10) are of order 0 in ϕ $(= \sqrt{\lambda^2+\mu^2})$, and $(\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})(\lambda \boldsymbol{c}^{\mathrm{T}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{A}_0^{\mathrm{I}} \boldsymbol{b})$ is a term of a higher order. So, we obtain (3.11).

Appendix C. By (5.10) and (5.11) we have

$$\begin{aligned} p &\approx p_0 + \lambda^2 \Bigg(-\sum_n \boldsymbol{x}_0^{\mathrm{T}} [\boldsymbol{D}_1]_{0n} [\boldsymbol{D}_0^{\mathrm{I}}]_{nn} [\boldsymbol{D}_1]_{n0} \boldsymbol{x}_0 \\ &\qquad + \lambda \sum_{m,n} \boldsymbol{x}_0^{\mathrm{T}} [\boldsymbol{D}_1]_{0m} [\boldsymbol{D}_0^{\mathrm{I}}]_{mm} [\boldsymbol{D}_1]_{mn} [\boldsymbol{D}_0^{\mathrm{I}}]_{nn} [\boldsymbol{D}_1]_{n0} \boldsymbol{x}_0 \Bigg) \\ &= p_0 + \lambda^2 \Bigg(-\sum_n |d_n|^2 \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{A}_1 \boldsymbol{x}_0 \\ &\qquad + \lambda \sum_{m,n} d_{-m} d_{m-n} d_n \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ms\boldsymbol{I})^{-1} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{A}_1 \boldsymbol{x}_0 \Bigg) \\ &\approx p_0 + \lambda^2 \Bigg(-\sum_n |d_n|^2 \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{b} \end{aligned} \tag{C.1}$$

$$+ \sum_{m,n} d_{-m} d_{m-n} d_n \lambda \boldsymbol{c}^{\mathrm{T}} (\boldsymbol{A}_0 - \mathrm{i}ms\boldsymbol{I})^{-1} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{b} \Bigg).$$

(The "splitting" trick (see the last paragraph of section 3) has been used in the last step of (C.1).) We calculate (C.1) term by term thus:

$$\begin{aligned} & -\sum_n |d_n|^2 \boldsymbol{x}_0^{\mathrm{T}} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{b} \\ &= R \sum_n |d_n|^2 \sum_m \frac{(2m\mu + \lambda^2 + \mu^2) m^2 |b_m|^2 (m^2 - \lambda^2 - \mu^2)}{(\lambda^2 + \mu^2)(m^2 + 2m\mu + insR)(m^2 + 2m\mu + \lambda^2 + \mu^2)} \\ &= \frac{R}{\lambda^2 + \mu^2} \sum_{m,n} m^4 |b_m|^2 |d_n|^2 \frac{(\lambda^2 + \mu^2)(m^4 + n^2 s^2 R^2) - 8\mu^2 m^4}{(m^4 + n^2 s^2 R^2)^2} + O(\phi^4) \\ &= \frac{R}{1 + k^2} \sum_{m,n} m^4 |b_m|^2 |d_n|^2 \frac{(1 + k^2)(m^4 + n^2 s^2 R^2) - 8k^2 m^4}{(m^4 + n^2 s^2 R^2)^2} + O(\phi^4); \end{aligned} \tag{C.2}$$

$$\begin{aligned} & \sum_{m,n} d_{-m} d_{m-n} d_n \lambda \boldsymbol{c}^{\mathrm{T}} (\boldsymbol{A}_0 - ims\boldsymbol{I})^{-1} \boldsymbol{A}_1 (\boldsymbol{A}_0 - \mathrm{i}ns\boldsymbol{I})^{-1} \boldsymbol{b} \\ &= \frac{2\lambda\mu R^2}{\lambda^2 + \mu^2} \sum_{m,n} d_{-m} d_{m-n} d_n \\ &\quad \times \sum_{j,l} \frac{j^2(\lambda^2 + \mu^2 + 2l\mu + 2jl - j^2)(l^2 - \lambda^2 - \mu^2)(j - l) l b_{-j} b_{j-l} b_l}{(j^2 + 2j\mu + \mathrm{i}msR)(l^2 + 2l\mu + \mathrm{i}nsR)[\lambda^2 + (j + \mu)^2][\lambda^2 + (l + \mu)^2]} \\ &= \frac{2\lambda\mu R^2}{\lambda^2 + \mu^2} \sum_{m,n} d_{-m} d_{m-n} d_n \sum_{j,l} \frac{jl(2l - j)(j - l) b_{-j} b_{j-l} b_l}{(j^2 + \mathrm{i}msR)(l^2 + \mathrm{i}nsR)} + O(\phi^4) \\ &= \frac{2\lambda\mu R^2}{\lambda^2 + \mu^2} \sum_{j,l,m,n} (b_{-j} b_{j-l} b_l)(d_{-m} d_{m-n} d_n) \frac{jl(2l - j)(j - l)(j^2 l^2 - mns^2 R^2)}{(j^4 + m^2 s^2 R^2)(l^4 + n^2 s^2 R^2)} + O(\phi^4) \\ &= \frac{2kR^2}{1 + k^2} \sum_{j,l,m,n} (b_{-j} b_{j-l} b_l)(d_{-m} d_{m-n} d_n) \frac{jl(2l - j)(j - l)(j^2 l^2 - mns^2 R^2)}{(j^4 + m^2 s^2 R^2)(l^4 + n^2 s^2 R^2)} + O(\phi^4). \end{aligned} \tag{C.3}$$

The result can be cast into the "eddy-viscosity" form (5.13):

$$\Re(p) = -D\phi^2 + \mathrm{O}(\phi^4). \tag{C.4}$$

(For the definition of D, see (5.14)–(5.17).)

Appendix D. From (5.25) and (5.26), R_1 and k_1 of (5.28) and (5.29) satisfy the linear equations

$$\left(a_0 + \frac{3}{2} R_0 k_0 c_0\right) R_1 + \left(2k_0^2 + \frac{1}{2k_0} R_0^3 c_0\right) k_1 = R_0^4 (a_1 - R_0 k_0 c_1), \tag{D.1}$$

$$a_0(1 + 7k_0^2) R_1 + (2 + 6k_0^2 + 7a_0 R_0^2) k_1 = a_1 R_0^4 (1 + 15k_0^2). \tag{D.2}$$

Here, a_0, a_1, c_0, and c_1 are the coefficients of the expansion of f_1, f_2, and f_3 given by (5.15)–(5.17):

$$f_1 = a_0 - a_1 R_0^2 s^2 + O(s^4), \tag{D.3}$$

$$f_2 = a_0 - 2a_1 R_0^2 s^2 + O(s^4), \tag{D.4}$$

$$f_3 = c_0 + c_1 R_0^2 s^2 + O(s^4), \tag{D.5}$$

so

$$a_0 = \sum_m |b_m|^2 \sum_m |d_n|^2, \tag{D.6}$$

$$a_1 = \sum_{m\neq 0} \frac{|b_m|^2}{m^4} \sum_n n^2 |d_n|^2, \tag{D.7}$$

$$c_0 = \sum_{j,l\neq 0} (b_{-j} b_{j-l} b_l) \frac{(2l-j)(j-l)}{jl} \sum_{m,n} (d_{-m} d_{m-n} d_n), \tag{D.8}$$

$$c_1 = \sum_{\substack{j,l,m,n \\ (j,l\neq 0)}} [(b_{-j} b_{j-l} b_l)(d_{-m} d_{m-n} d_n)] \frac{(2l-j)(j-l)}{jl} \left(\frac{m^2}{j^4} + \frac{n^2}{l^4} + \frac{mn}{j^2 l^2} \right). \tag{D.9}$$

Hence, R_1 and k_1 are readily found when R_0 and k_0 are known.

Appendix E. We want to find the critical frequency s_c and also the approximate expressions of $R(s)$ and $k(s)$ at $s \uparrow s_c$. By the definition of s_c (see section 5), $R \to \infty$ as $s \uparrow s_c$. So we take $1/R$ as the perturbation parameter and let

$$k = k_0 + \frac{k_1}{R} + O\left(\frac{1}{R^2}\right), \tag{E.1}$$

$$s^2 = s_c^2 - \frac{s_1^2}{R^2} + O\left(\frac{1}{R^4}\right). \tag{E.2}$$

The quantities f_1, f_2, and f_3 can be expanded in the perturbed form as follows:

$$f_1 = \frac{1}{s_c^2 R^2} f_1^{(1)} - \frac{1}{s_c^4 R^4} f_1^{(2)} + \frac{s_1^2}{s_c^4 R^4} R_1^{(1)} + O\left(\frac{1}{R^6}\right), \tag{E.3}$$

$$f_2 = \frac{1}{s_c^4 R^4} f_2^{(1)} + O\left(\frac{1}{R^6}\right), \tag{E.4}$$

$$f_3 = \frac{-1}{s_c^2 R^2} f_3^{(1)} + \frac{1}{s_c^4 R^4} f_3^{(2)} + \frac{s_1^2}{s_c^4 R^4} f_3^{(1)} + O\left(\frac{1}{R^6}\right), \tag{E.5}$$

where

$$f_1^{(1)} = \sum_m m^4 |b_m|^2 \sum_{n\neq 0} \frac{|d_n|^2}{n^2}, \tag{E.6}$$

$$f_1^{(2)} = f_2^{(1)} = \sum_m m^8 |b_m|^2 \sum_{n\neq 0} \frac{|d_n|^2}{n^4}, \tag{E.7}$$

$$f_3^{(1)} = \sum_{j,l} jl(2l-j)(j-l)(b_{-j} b_{j-l} b_l) \sum_{\substack{m,n \\ (m,n\neq 0)}} \frac{d_{-m} d_{m-n} d_n}{mn}, \tag{E.8}$$

(E.9)

$$f_3^{(2)} = \sum_{\substack{j,l,m,n\\(m,n\neq 0)}} \frac{jl(2l-j)(j-l)(mnj^2l^2+n^2j^4+m^2l^4)(b_{-j}b_{j-l}b_l)(d_{-m}d_{m-n}d_n)}{m^3n^3}.$$

In fact, the last sum in (E.8) is zero. Indeed, let

$$S = \sum_{\substack{m,n\\(m,n\neq 0)}} \frac{d_{-m}d_{m-n}d_n}{mn}. \tag{E.10}$$

S can be written in a symmetric form: denote $\hat{m} = -m$, $\hat{n} = n$, and $\hat{k} = m - n$ so that $\hat{m} + \hat{n} + \hat{k} = 0$. Omitting the hats, we have

$$S = \sum_{\substack{m+n+k=0\\(m,n\neq 0)}} \frac{-d_m d_n d_k}{mn}. \tag{E.11}$$

Exchanging the indexes n and k here, we have

$$S = \sum_{\substack{m+n+k=0\\(m,k\neq 0)}} \frac{-d_m d_n d_k}{mk}. \tag{E.12}$$

Similarly,

$$S = \sum_{\substack{m+n+k=0\\(n,k\neq 0)}} \frac{-d_m d_n d_k}{nk}. \tag{E.13}$$

Hence, taking the average of (E.11)–(E.13), we can write

$$\begin{aligned} S &= -\frac{1}{3} \sum_{\substack{m+n+k=0\\(m,n,k\neq 0)}} d_m d_n d_k \left(\frac{1}{mn} + \frac{1}{mk} + \frac{1}{nk}\right) \\ &= -\frac{1}{3} \sum_{\substack{m+n+k=0\\(m,n,k\neq 0)}} d_m d_n d_k \frac{m+n+k}{mnk} = 0, \end{aligned} \tag{E.14}$$

where the last sum is clearly zero since each term of it contains the zero factor $(m + n + k)$. Thus,

$$f_3^{(1)} = 0. \tag{E.15}$$

The substitution of the asymptotic expressions (E.1)–(E.5) into (5.18) and (5.19) yields the equations for k_0 and s_c:

$$(1 + k_0^2)\left[(1 + k_0^2) - \frac{1}{s_c^2} f_1^{(1)}\right] = 0, \tag{E.16}$$

$$k_0 \left[2(1 + k_0^2) - \frac{1}{s_c^2} f_1^{(1)}\right] = 0. \tag{E.17}$$

The unique solution is

$$k_0 = 0, \tag{E.18}$$
$$s_c = \sqrt{f_1^{(1)}}.$$

The equations to determine k_1 and s_1^2 (coming from the order $\frac{1}{R^2}$) are

$$2k_1^2 - \frac{k_1^2}{s_c^2} f_1^{(1)} + \frac{1}{s_c^4} f_1^{(2)} - \frac{s_1^2}{s_c^4} f_1^{(1)} - \frac{2k_1}{s_c^4} f_3^{(2)} = 0, \tag{E.19}$$

$$2k_1 - \frac{k_1}{s_c^2} f_1^{(1)} - \frac{1}{s_c^4} f_3^{(2)} = 0. \tag{E.20}$$

Using the second equation of (E.18), equations (E.19) and (E.20) reduce to

$$k_1 - \frac{1}{s_c^4} f_3^{(2)} = 0, \tag{E.21}$$

$$k_1^2 - \frac{1}{s_c^4} f_1^{(2)} + \frac{s_1^2}{s_c^2} = 0. \tag{E.22}$$

It is obvious that the solutions of (E.21) and (E.22) are

$$k_1 = \frac{1}{s_c^4} f_3^{(2)}, \tag{E.23}$$

$$s_1 = \frac{1}{s_c} \left[f_1^{(2)} - \frac{1}{s_c^4} (f_3^{(2)})^2 \right]^{\frac{1}{2}}. \tag{E.24}$$

Alternatively, from (E.1) and (E.2), we can represent R and k as the following asymptotic dependencies on s:

$$R = \frac{s_1}{\sqrt{s_c^2 - s^2}} \qquad (s \uparrow s_c), \tag{E.25}$$

$$k = \frac{k_1 \sqrt{s_c^2 - s^2}}{s_1} \qquad (s \uparrow s_c). \tag{E.26}$$

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