On evolution equations for thin films flowing down solid surfaces

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A wavy free-surface flow of a viscous film down a cylinder is considered. It is shown that if the cylinder radius is large, as compared to the film thickness, the long-wave perturbation approach yields a rather simple evolution equation. This nonlinear equation is similar to the well-known Benney equation of planar films, and becomes exactly the latter in the limit of infinite radius. Thus it is the annular-case analog—which was missing in the literature—of the Benney equation. It is argued that under conditions implicitly implied in their derivation, the Benney-type equations are not uniformly valid for large times. However, both the new and Benney equations are important heuristically—as sources of other, simpler, equations which, in certain domains of system parameters, are valid for all time. Also, the new equation of annular films is important as a qualitative model incorporating all significant physical factors.

I. INTRODUCTION

Flows of liquid films on solid surfaces are important in industry and nature, and have been drawing significant research interest for many years. In theoretical research, planar films were studied first (a recent review and further references can be found, e.g., in Ref. 1). Benney[2] initiated a long-wave perturbation approach which led to the nonlinear equation [(9) in Ref. 1] for film thickness. The equation has the form of a power series in the perturbation parameter α, the ratio of the average film thickness to the wavelength. Usually, only the two first terms of the series are retained; this results in what is sometimes called the Benney equation.

It contains the effects of inertia; the stabilizing action of surface tension is also included if the latter is sufficiently large (a certain Weber number, proportional to the surface tension constant, is assumed to be of order α^{-2}). Modifications which account for other factors, such as molecular forces and nonisothermal effects, have been obtained (e.g., Ref. 3); those have principally the same two-perturbation-orders structure as the paradigmatic Benney equation. One would expect that, for annular films, there should exist an analogous equation. However, no equation of this kind is found in the literature.

Atherton and Homsy[4] and Lin and Liu[5] did apply the long-wave approach of Benney to a film flow down a cylinder. They derived in principal the equation for a long-wave evolution of film thickness. However, the calculations are much more involved than those in the planar case (and were performed "with the aid of symbolic computation using REDUCE" in Ref. 4), and the coefficients of the resulting equation are unwieldy.

In the present paper, I show that there is a way to obtain for annular films a simple analog of the Benney equation. To this end, one should assume the radius of the cylinder to be sufficiently larger than the film thickness, namely, of the same order of magnitude as the large axial wavelength, or larger. I show that this assumption, combined with the long-wave perturbation approach, leads to a new evolution equation which (i) is rather simple, (ii) is quite similar to the Benney equation, and (iii) becomes exactly the latter in the limit of infinite radius of the cylinder. Thus, this equation is the missing annular-film counterpart to the (planar-film) Benney equation.

There appears to be another deficiency in the literature on the Benney equation which I attempt to make up here: I argue that the equations of this type, derived by a formal perturbation method, may not be uniformly valid in time. The reason is, roughly, that the long-wave perturbation parameter is based, in effect, on the wavelength of the *initial* state. However, the dissipative system evolves to attracting regimes which have an intrinsic characteristic length scale, determined by the *basic parameters*. Since the system "forgets" the initial conditions, the initial long-wave parameter may become irrelevant after some transient time. So, an uncritical acceptance of the long-wave equations may lead to wrong conclusions as to the long-time evolution of the film.

Having said that, I will argue that there are certain important roles (which were mentioned in the abstract) for the Benney-type equations to play in film flow studies.

The rest of the paper is structured as follows.

In the next section, the Navier–Stokes problem is formulated. The perturbation approach to that problem yields the evolution equation (29) in Sec. III. [Its three-dimensional generalization is given by Eq. (30).] In Sec. IV, I discuss the relations of both Eq. (29) and Benney's equation between themselves and to other evolution equations, along with the question of large-time validity for each of these equations.

The conclusions are summarized in Sec. V.

II. NAVIER–STOKES PROBLEM

The Navier–Stokes (NS) equations describing (for simplicity) an axially symmetric flow, with zero azimuthal velocity, of an incompressible fluid of density $\bar{\rho}$ (the overbar marks dimensional quantities) and viscosity $\bar{\mu}$ can be written in the following dimensionless form:[4,5,6,7]

$$r^{-1}(rv)_y+u_z=0, \quad (1)$$

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$$u_t+uu_z+vu_y=-p_z+R^{-1}[r^{-1}(ru_y)_y+u_{zz}]+2R^{-1}, \tag{2}$$

$$v_t+vv_y+uv_z=-p_y+R^{-1}\{[r^{-1}(rv)_y]_y+v_{zz}\}. \tag{3}$$

Here, the average film thickness $\bar{h}_0$ is the unit of length, e.g., for the downward coordinate $\bar{z}$ and the radial one $\bar{y}=\bar{r}-\bar{b}$, where $\bar{b}$ is the radius of the cylinder $y=0$. The interface velocity $\bar{U}=\bar{g}\bar{h}_0^2/(2\bar{\nu})$ of the (Nusselt's unstable equilibrium) flow of a film—which has a constant thickness $\bar{h}_0$ and kinematic viscosity $\bar{\nu}=\bar{\mu}/\bar{\rho}$—down a vertical *plane* under the action of gravity (acceleration $\bar{g}$) is the unit for the z component of the velocity, u, and the y component, v. Other units are based on $\bar{h}_0$ and $\bar{U}$: $\bar{h}_0/\bar{U}$ for time, t; $\bar{\rho}\bar{U}^2$ for the pressure p; and $\bar{\rho}\bar{U}^2\bar{h}_0$ for the surface tension γ. In (1)–(3), the subscripts denote the corresponding partial derivatives; and R is the Reynolds number,

$$R=\bar{U}\bar{h}_0/\bar{\nu}\ [=\bar{h}_0^3\bar{g}/(2\bar{\nu}^2)]. \tag{4}$$

The boundary conditions are as follows. The no-slip conditions at the solid cylinder surface are

$$u=0 \quad\text{and}\quad v=0 \quad (y=0). \tag{5}$$

The stress balance conditions at the free surface $y=h(z,t)$ are

$$(1-h_z^2)(u_y+v_z)+2h_z(v_y-u_z)=0, \tag{6}$$

$$p+R^{-1}(u_y+v_z)h_z-2R^{-1}v_y+\gamma(1+h_z^2)^{-1/2}\times[(1+h_z^2)^{-1}h_{zz}-r^{-1}]=0, \tag{7}$$

(the pressure of the ambient gas is neglected for simplicity).

Finally, the kinematic condition at the free surface is

$$h_t+uh_z-v=0 \quad (y=h). \tag{8}$$

III. LONG-WAVE LARGE-RADIUS PERTURBATION THEORY

Consider the case of a "large-radius" cylinder: $b\gg 1$. According to the linear theory (e.g., Ref. 5), the surface tension leads to capillary instability with its maximum at the length scales of order b. In the nonlinear long-wave theory, we assume that waves have (presumably, because the initial conditions are chosen to be such) a large length-scale l:

$$\partial_z\sim l^{-1}=\alpha\ll 1. \tag{9}$$

We rescale the axial coordinate and time: $\xi=z/l$ and $\tau=t/l$, so that

$$\partial_z=\alpha\partial_\xi,\quad \partial_t=\alpha\partial_\tau, \tag{10}$$

and then $\partial_\xi\sim 1$, $\partial_\tau\sim 1$, $\partial_y\sim 1$.

Let $\beta=l/b$, so that $b=\alpha^{-1}\beta^{-1}$; then

$$r=b+y=\alpha^{-1}\beta^{-1}+y;$$

$$r^{-1}=(b+y)^{-1}=\alpha(\beta)-\alpha^2(\beta^2y)+O(\alpha^3) \tag{11}$$

(where, for the last equality, the binomial theorem was used).

We will substitute the perturbation expansions in powers of α, such as

$$u=u^0+\alpha u^1+\cdots \tag{12}$$

[where $u^0=u^0(\xi,y,\tau)$, etc.; the superscript i marks the coefficient of the power α^i], and also relations (10) and (11), into Eqs. (1)–(8), so that the left-hand sides of (1)–(8) become power series in α.

From (1), $v_y\sim\alpha u_\xi$; therefore, the leading order of v is $O(\alpha)$, so that v can be written as follows:

$$v=\alpha(v^0+v^1\alpha^1+\cdots). \tag{13}$$

We assume $\tilde{\gamma}=\gamma\alpha^2\sim 1$ (and $R\sim 1$); then, differentiating (7) along the z direction, we obtain

$$p_\xi=p_\xi^0+p_\xi^1\alpha+\cdots, \tag{14}$$

where

$$p_\xi^0=-\tilde{\gamma}(h_{\xi\xi\xi}+\beta^2h_\xi). \tag{15}$$

The axial NS equation (2) with substitutions (12)–(14) becomes

$$\alpha^0(u_{yy}^0+2)+\alpha^1[-R(u_\tau^0+u^0u_\xi^0+v^0u_y^0+p_\xi^0)+\beta u_y^0+u_{yy}^1]+\cdots=0. \tag{16}$$

The coefficient of α^i must be zero, for each i. In particular, for $i=0$,

$$u_{yy}^0=-2. \tag{17}$$

The boundary conditions are given by the leading orders of (5) and (6):

$$u^0=0\ (y=0);\quad u_y^0=0\ (y=h). \tag{18}$$

The solution of this simple problem, (17) and (18), is

$$u^0=-y^2+2hy. \tag{19}$$

The equation for v^0 follows from the continuity equation (1) in the leading order, $O(\alpha)$:

$$v_y^0=-u_\xi^0,\ =-2h_\xi y, \tag{20}$$

where the last equality follows from (19). With the boundary condition (5), $v^0=0$ at $y=0$, the solution is

$$v^0=-h_\xi y^2. \tag{21}$$

The substitution of (19) and (21) with $y=h$ into the leading order α of (8) written as

$$\alpha(h_\tau+u^0h_\xi-v^0)+\alpha^2(u^1h_\xi-v^1)+\cdots=0 \tag{22}$$

yields

$$h_\tau+2h^2h_\xi=0, \tag{23}$$

the leading-order evolution equation for the film thickness $h(\xi,\tau)$. To find the coefficient of the next-order correction in (22), $u^1h_\xi-v^1$, we need the expressions of u^1 and v^1 in terms of h. These are found from the following coefficient equations. The axial NS equation (2) in order α gives [see (16)]

$$u_{yy}^1=-\beta u_y^0+R[u_\tau^0+u^0u_\xi^0+v^0u_y^0+p_\xi^0],$$

$$=2Rhh_\xi y^2+2y(\beta-2Rh^2h_\xi)+Rp_\xi^0-2h\beta, \qquad (24)$$

where the last equality follows from the expressions (19) and (21) for u^0 and v^0.

The boundary conditions follow from (5) and (6) in order α:

$$u^1=0\ (y=0); \quad u_y^1=0\ (y=h). \qquad (24')$$

As a result,

$$u^1=6^{-1}Rhh_\xi y^4+3^{-1}(\beta-2Rh^2h_\xi)y^3+(2^{-1}Rp_\xi^0-h\beta)y^2 + [R(4h^4h_\xi/3-hp_\xi^0)+h^2\beta]y. \qquad (25)$$

This solution is readily verified by its direct substitution into the problem equations (24) and (24'). Similarly, the continuity equation (1),

$$\alpha(v_y^0+u_\xi^0)+\alpha^2(v_y^1+\beta v^0+u_\xi^1)+\cdots=0,$$

gives, in the order α^2, the equation for v^1,

$$v_y^1=-\beta v^0-u_\xi^1. \qquad (26)$$

The solution which satisfies the boundary condition following from (5),

$$v^1=0\ (y=0), \qquad (27)$$

is

$$v^1=-30^{-1}R(hh_\xi)_\xi y^5+6^{-1}R(h^2h_\xi)_\xi y^4-6^{-1} \times(Rp_\xi^0-4h\beta)_\xi y^3+[6^{-1}R(3hp_\xi^0-4h^4h_\xi)_\xi-hh_\xi\beta]y^2, \qquad (28)$$

which is not difficult to verify by direct substitution of (28), (21), and (25) into the problem equations (26) and (27).

Finally, by putting $y=h$ in (25) and (28), substituting those expressions into (22), and expressing p_ξ^0 by (15), we find the evolution equation for h (ξ,τ):

$$h_\tau+2h^2h_\xi+\alpha\left(\frac{1}{3}R\tilde{\gamma}h^3(\beta^2h_\xi+h_{\xi\xi\xi}) + \frac{8}{15}Rh^6h_\xi+\frac{1}{6}\beta h^4\right)_\xi+\cdots=0. \qquad (29)$$

Similarly, an equation can be obtained for the most general, nonaxisymmetric, disturbance which may depend on the azimuthal coordinate θ in addition to the axial one, ξ:

$$h_\tau+2h^2h_\xi+\alpha\left(\frac{8}{15}R(h^6h_\xi)_\xi+\frac{1}{6}\beta(h^4)_\xi + \frac{2}{3}\tilde{W}\nabla[h^3\nabla(\beta^2h+\nabla^2h)]\right)=0, \qquad (30)$$

where $\tilde{W}=\frac{1}{2}R\tilde{\gamma}=\frac{1}{2}R\gamma\alpha^2$, a modified Weber number; $\nabla=\hat{z}\partial_\xi+\hat{\theta}\delta\partial_\zeta$; $\hat{z}$ and $\hat{\theta}$ are unit vectors in the corresponding directions; and $\zeta=\delta\beta^{-1}\theta$, a rescaled azimuthal coordinate, which implies that the axial wavelength is δ times greater than the azimuthal one.

Clearly, for a θ-independent disturbance, $\partial_\zeta=0$, $\nabla=\hat{z}\partial_\xi$ and, hence, (30) returns to the form (29).

IV. DISCUSSION

If the cylinder radius b tends to infinity, β goes to zero. Equation (29) with $\beta=0$ is

$$h_\tau+2h^2h_\xi+\alpha\left(\frac{8}{15}Rh^6h_\xi+\frac{2}{3}\tilde{W}h^3h_{\xi\xi\xi}\right)_\xi=0, \qquad (31)$$

where $\tilde{W}=\frac{1}{2}R\tilde{\gamma}=W\alpha^2$ and $W=\frac{1}{2}R\gamma=\bar{\gamma}/(\bar{\rho}\bar{g}\bar{h}_0^2)$, the Weber number.

Equation (31) pertains to the flow down a vertical *plane*. The general equation for the flow down a plane inclined at an angle φ to horizontal was derived[8] by Benney's long-wave approach and is called the Benney equation. (Numerical simulations of this equation were reported, e.g., in Refs. 9–11.) In Ref. 1, it is written [Eq. (9) there] in the form which includes terms of order α^2 as well:

$$h_\tau+A(h)h_\xi+\alpha[B(h)h_\xi+C(h)h_{\xi\xi\xi}]_\xi + \alpha^2[E(h)h_{\xi\xi}+F(h)h_{\xi\xi\xi\xi}+\cdots]_\xi+O(\alpha^3)=0 \qquad (32)$$

(where we omitted some terms of order α^2 which do not play any role below). The coefficients are given in Ref. 1 as

$$A(h)=2h^2 \qquad (33)$$

and, if specified for the case of vertical plane ($\varphi=\pi/2$, i.e., $\cot\varphi=0$),

$$B(h)=\frac{8}{15}Rh^6, \quad C(h)=\frac{2}{3}\alpha^2Wh^3, \qquad (34)$$

so that truncation of (32) at the order α gives (31).

Equation (31) is sometimes used to study waves of large amplitude, $(h-1)\sim 1$ (e.g., Ref. 11). Since it incorporates the physically relevant effects of both inertia and surface tension, (31) may be a qualitatively good model for the film flow (indeed, some of its consequences compare favorably with experiments; however, some other predictions fail[12]).

But the question of whether there is a sound asymptotic basis to expect that, for the case $(h-1)\sim 1$, (31) can provide *quantitatively* good approximation uniformly in time needs to be discussed. Indeed, (31) lumps together terms of different asymptotic orders. The leading order equation is just

$$h_\tau+A(h)h_\xi=0. \qquad (35)$$

It is well known that this equation describes steepening of the forward faces of the waveform which leads to eventual breaking of the waves after a finite time. To stop this process, some of the order-α terms must become comparable to the leading order terms; otherwise they would give only small corrections to the leading order breaking solution, and thus could not prevent the wave breaking (whereas, in reality—as evidenced by numerical experients[10]—the steepening of the Benney waves comes to an end, and they never break). The B and C terms of (32) can become large, e.g., because of effective scale contraction due to the steepening, so that

$$\sigma=\frac{\partial}{\partial\xi}\gtrsim 1 \qquad (36)$$

in (32). Among the $O(\alpha)$ terms, the B term of (32) cannot be neglected, since the only other one is the C term (due to the surface tension) which would cause the wave amplitude to decay. Hence, to have a wave which persists in time, the B term should be comparable to the leading-order A term, i.e., (B term/A term) $\gtrsim 1$. The estimate of this ratio is, by using (33), (34), and (36),

$$1 \lesssim \frac{B\ \text{term}}{A\ \text{term}} \sim \frac{\alpha R h^6 h \sigma^2}{h^2 h \sigma} = R h^4 \alpha \sigma \tag{37}$$

[the ξ derivatives are estimated as $h_{\xi\xi} \sim \sigma^2 h$, etc.; see (36) and recall that $(h-1) \sim 1$].

However, by using the expressions[1] for the $O(\alpha^2)$ coefficients E and F of (32), namely,

$$E(h) = \frac{32}{63} R^2 h^{10} + 2h^4; \quad F(h) = \frac{40}{63} \alpha^2 R W h^7, \tag{38}$$

we obtain the following estimate:

$$\frac{E\ \text{term}}{B\ \text{term}} \gtrsim \frac{\alpha^2 \sigma^3 R^2 h^{10} h}{\alpha \sigma^2 R h^6 h} = R h^4 \alpha \sigma \gtrsim 1, \tag{39}$$

where the last inequality follows from (37). Thus there are terms of order α^2 in (32) which cannot be neglected. The same applies to $O(\alpha^3)$ terms, etc. So, the perturbation theory breaks down. There is no single equation which can approximate the evolution uniformly in time. The leading-order equation (35) is good for a limited time, but after it breaks down, turning as a remedy to Eq. (31) is not justified. [Of course, before that breakdown, there may be a limited interval of time during which (35) yields a somewhat better approximation than (31).] Equation (31) may still be a qualitatively reasonable model, even for large times, since it captures all physically important factors, such as surface tension and inertia; but one should not expect its solutions to be quantitatively good *time-uniform* approximations to the exact solutions of the film problem.

It is easy to see that the same arguments hold for the general Benney equation [(9) in Ref. 1], that is for the case of a nonvertical plane, cot $\varphi \neq 0$. Similar conclusions are reached for the cylinder case, Eq. (29). All these equations are not uniformly valid in time in their original form implying large amplitude waves, $(h-1) \sim 1$. (Another scenario in which the Benney equation should become invalid after a finite time was indicated in Refs. 11 and 10: For some initial conditions, the solution amplitudes can grow to infinity, and the underlying assumption of small slope is eventually violated. However, it appears that the equation becomes invalid even earlier—via the steepening mechanism suggested above.) However, from these Benney-type equations others can be obtained which *are* likely to be valid for all time. For example, if the wave amplitude is of order α, that is

$$h - 1 = \alpha \eta, \quad (\eta \sim 1), \tag{40}$$

(31) becomes an equation for η via the substitution $h = 1 + \alpha\eta$. In a moving reference frame—changing ξ to $x = \xi - 2\tau$, so that $\eta_\tau \to (\eta_\tau - 2\eta_x)$ and $\eta_\xi = \eta_x$—Eq. (31) becomes

$$\alpha \eta_\tau + 4\alpha^2 \eta \eta_x + \alpha \left(\frac{8}{15} R \alpha \eta_x + \frac{2}{3} \tilde{W} \alpha \eta_{xxx} \right)_x = 0.$$

After rescaling the time variable, $T = \alpha\tau$, all the terms contain the factor α^2, and, hence (as was first derived from the Benney equation in Ref. 13),

$$\eta_T + 4\eta\eta_x + \frac{8}{15} R \eta_{xx} + \frac{2}{3} \tilde{W} \eta_{xxxx} = 0. \tag{41}$$

The above change of reference frame is not necessary for the derivation of (41). If we remain in the original frame, there is a "fast-time" change of η at a fixed location, due to the uniform translation of the wave pattern past the observer. So, η should depend both on the fast time τ and on the "slow time" $T = \alpha\tau$, so that $\partial_\tau \to \partial_\tau + \alpha \partial_T$. Then the leading order of (31) is $O(\alpha)$: $\alpha\eta_\tau + 2\alpha\eta_\xi = 0$. The equation $\eta_\tau + 2\eta_\xi = 0$, is, as was mentioned above, of purely kinematic character: it describes the uniform translation of the waveform. This contrasts with the leading order (35) of the large-amplitude case, which describes steepening and breaking of waves. In order α^2, we obtain (41), which describes the intrinsic dynamics of the wave pattern. The "benign" leading-order behavior in the small amplitude case is the necessary basis for the time uniformity of the dynamical description (41).

Equation (41) is, so to say, "asymptotically consistent:" it does not lump together terms of different asymptotic orders. It is well known from many computational experiments (e.g., Ref. 14) that solutions of the Kuramoto–Sivashinsky type equation (41) on extended spatial intervals are attracted to regimes of chaotic waves whose characteristic amplitudes, as well as length and time scales, are of magnitude-order one if $R \sim 1$ and $\tilde{W} \sim 1$. This estimate of scales also follows from the pairwise balance of terms in (41); a simple dynamical mechanism enforcing that balance was suggested in Ref. 15. The estimate $\eta \sim 1$ for solutions of (41) is in accordance with the small-amplitude assumption (40).

Both the numerical experiments[14] and the theoretical arguments[15] show that, starting from an arbitrary small-amplitude disturbance whose length scale may be large or small, the film evolves to a state with characteristic wavelengths which (assuming $R \lesssim 1$ and $\tilde{W} \gtrsim 1$) are large (as compared to the film thickness) by the factor of order $W^{1/2} R^{-1/2} \gg 1$, and whose amplitudes are small, $\sim R^{3/2} W^{-1} \ll 1$. Equation (41) should give a quantitatively good approximation for the low-amplitude evolution of the film: It is easy to check that, with the above estimates of characteristic quantities, the terms of the full Navier–Stokes problem which are effectively discarded [in reducing the evolution description to the single equation (41)] are estimated to be much smaller than the terms retained. This is not so, for example, for larger values of R, when the amplitude is estimated to be of order 1, so that (40), and therefore (41), is no longer valid. But in those cases (31) is not good either, by the same reasoning: the NS terms which have to be left out in its derivation are not small.

Returning to the annular flow, Eqs. (29) or (30), an asymptotically consistent equation for small-amplitude regimes is obtained in exact analogy with deducing (41) from (31). That yields

$$\eta_T+\frac{2}{3}\beta\eta_x+4\eta\eta_x+\frac{8}{15}R\eta_{xx}+\frac{R}{3}\beta^2\nabla^2\eta+\frac{2\widetilde{W}}{3}\nabla^4\eta=0. \tag{42}$$

In Ref. 6, this equation was derived directly from NS, as a leading-order coefficient equation in a formal single-parameter perturbation scheme, by putting $\alpha=(2\widetilde{W}/R)^{-1/2}$ (note that $\widetilde{W}=R/2$ as a result) and looking for solutions in the form of power series in α; in particular, $h=1+\alpha\eta+\cdots$. [Note that some signs in Eq. (35) of Ref. 6 are opposite to those in (42), due to the different choice of the positive direction on the vertical axis; the signs in (42) agree with Ref. 16.]

But, in contrast to the planar case, a time-uniform equation describing large waves, $(h-1)\sim 1$, can be deduced from the annular-film equation (29). Namely, let R and $\widetilde{\gamma}$ be such that

$$\alpha R\widetilde{\gamma}=\widetilde{S}\sim 1 \tag{43}$$

(which means $W\sim\alpha^{-3}$, rather than the usual $W\sim\alpha^{-2}$). Then the first term in brackets of (29) becomes of order one (assuming $h\sim 1$, $h_\xi\sim 1$, etc.) and the leading-order equation is (with $\beta=1$, i.e., $\alpha=b^{-1}$)

$$h_\tau+2h^2h_\xi+\tfrac{1}{3}\widetilde{S}[h^3(h_\xi+h_{\xi\xi\xi})]_\xi=0. \tag{44}$$

This equation was derived in Ref. 7 directly from the NS problem, by using a multiparameter perturbation approach[17] that yields simultaneously a domain in the parameter space for the validity of the theory: If S has the order of magnitude one, then $\alpha=b^{-1}\ll 1$ and $R\alpha\ll 1$ are the conditions of applicability, for the case under consideration. Some scaling dependencies[7] which follow from (44) are in excellent agreement with experiment.[18]

Equation (44) is the only known time-uniform[19] equation for large-amplitude waves on films with nonzero average flow (for films with no mean transport, there are evolution equations which can handle large amplitudes, e.g., Ref. 20 for the planar case and Ref. 21 for the cylindrical one). The reason for this unique opportunity in the cylindrical case is the fact that surface tension can provide, along with the usual stabilizing effect due to the longitudinal curvature, a destabilizing action, induced by the transverse curvature which is absent in the planar case. Therefore, there is no need for a large inertia (to grow the large waves), which would break up the perturbation theory responsible for reducing the problem to a single evolution equation. [For numerical simulations of (44) which reveal some interesting evolution processes, see Ref. 19.]

Although the "parental" equation (29) itself is not strictly consistent, it can be studied as a model, similar to the investigation[11] of its planar analog, the Benney equation (31). [Note that (31)—in a slightly generalized version that includes nonvertical plane case—was written in Ref. 11 in the form which is obtained by returning to our original unscaled quantities, t, z, and W: $h_t+2h^2h_z+\frac{8}{15}(Rh^6h_z+\frac{2}{3}Wh_{zzz})_z=0$. In this form, the fact that the terms in parentheses are—because $\partial_t\sim\alpha\ll 1$, $\partial_z\sim\alpha$, etc.—of a higher asymptotic order than the other terms is somewhat less obvious, but true nevertheless.]

Since (29) and (31) are very similar, all the conclusions obtained for (31)—such as the existence of solitary waves and self-similar solutions which reach singularity in a finite time (see Refs. 10 and 11)—should hold for Eq. (29). However, since both (29) and (31) are only model equations, their consequences should be approached with caution as far as their relevancy to the physical reality is concerned.

V. CONCLUDING REMARKS

The reason why previous attempts to apply the long-wave perturbation method to annular film flows did not produce a manageable evolution equation [which is (29) above] is just the fact that the assumption of large radius was not utilized.

Although both Eq. (29) and its well-known inclined-plane analogue, the Benney equation (31), lump together terms of different asymptotic orders, and therefore, typically, either retain terms which can be neglected or neglect terms which eventually are not small, they are useful heuristically, as generators of consistent equations which may be uniformly valid in time. One of the latter describes large-amplitude regimes for the annular flow, whereas such waves in inclined-plane film flows are not reducible to any single evolution equation. In the parametric domains where only small waves develop, such time uniform equations—which can be formally obtained from the Benney-type equations—are of the Kuramoto–Sivashinsky type.

Since the new equation (29) includes the relevant physical factors, it can be studied as a model, on a par with its planar counterpart (31), and similar results—solitary waves, explosive self-similar solutions, etc.—can be obtained. However, the model character of these Benney-type equations should be remembered in comparing their results with experiments.

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